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Two Commutativity Results for Semiprime Rings*

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Throughout this paper, R will represent an associative ring (may be without unity) with center $Z(R)$. Given x, y in R, we set $[x, y]=xy-yx$ as usual.

Quadri, Ashraf and Khan [6] proved that a semiprime ring R satisfying $(xy)^2-xy\in Z(R)$ for all $x, y\in R$ is commutative. In this direction we prove the following

Theorem 1 Let R be a semiprime ring satisfying

(C) $$(x^m y)^n - x^m y\in Z(R)$$

for all $x, y\in R$, where m, n are fixed positive integers, $n>1$, then R is commutative.

In [2] Gupta has proved that if a semiprime ring R with unity satisfies

(1) $[x^n, y]-[x, y^n]\in Z(R)$ (2) $[x^{n+1}, y]-[x, y^{n+1}]\in Z(R)$

for all $x, y\in R$ and a fixed integer $n>1$, then R is commutative. In this direction we prove the following

Theorem 2 Let R be a semiprime ring satisfying

(P) $$[x^m, y^n]^t-[x, y^s]\in Z(R)$$

for all $x, y\in R$, where m, n, s, and t are fixed positive integers such that $(m+n)t=s+1$ and $mt>1$, then R is commutative.

For the proofs of above theorems, we need the following lemmas.

Lemma 1[2] Let R be a semiprime ring, $0\neq a\in Z(R)$ and $x\in R$. If $ax\in Z(R)$, then $x\in Z(R)$.

Lemma 2[5] If a ring R has a nonzero right ideal A which is nil of bounded index, then R has a nonzero nilpotent ideal.

Lemma 3 Let R be a prime ring satisfying (C), then R has no nonzero nilpotent elements.

Proof Let $a\in R$ such that $a^2=0$. By hypothesis we have $-(ax)^m a=[(ax)^m a]^n-(ax)^m a\in Z(R)$ for all $x\in R$. Thus, $(ax)^{m+2}=(ax)^m axax=xa(ax)^m ax=0$. If $aR\neq 0$, then R has a nonzero nilpotent ideal by Lemma 2, which contradicts to the fact

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that R is prime. Thus, $aR=0$, and hence $aRa=0$. This implies that $a=0$.

Lemma 4 Let R be a division ring satisfying $x^n=x$ for all $x\in Z(R)$, where $n>1$ is a fixed integer, then $|Z(R)|\leq n$.

Proof Since $Z(R)$ is a field, $x^n=x$ has at most n solutions.

Proof of Theorem 1 As is well known, R is a subdirect sum of prime rings R_a each of which as a homomorphic image of R satisfies (C). Then it suffices to show that if R is a prime ring satisfying (C), then R is commutative.

If there exists an element c in $Z(R)$ such that $c^n-c\neq 0$. By hypothesis we have $c^n(x^my)^n-c(x^my)\in Z(R)$ for all $x, y\in R$. But $c^n(x^my)^n-c^n(x^my)\in Z(R)$. Then $(c^n-c)(x^my)\in Z(R)$. Thus, by Lemma 1 we get $x^my\in Z(R)$ for all $x, y\in R$. Now, by Lemma 3 and a result of Herstein [3], R is commutative.

If $c^n=c$ for all $c\in Z(R)$, then we have

$$(x^{(m+1)n}-x^{m+1})^n=x^{(m+1)n}-x^{m+1} \text{ for all } x\in R. \text{ That is } x^{m+1}f(x)=x^{m+1},$$

where $f(x)$ is a polynomial with integral coefficients, $f(0)=0$. Thus, R satisfies a polynomial identity and $x^{m+1}=0$ for all $x\in J(R)$, the Jacobson radical of R. Hence, by Lemma 3 $J(R)=0$. Then it suffices to show that a primitive ring R satisfying (C) is commutative.

Suppose that R is a division ring, then by [4, Theorem 1] R is finite-dimensional over $Z(R)$. By Lemma 4 we have $|Z(R)|\leq n$. Then R is a finite division ring and therefore a field.

Suppose now that R is a primitive ring, if it is not a division ring, then, the complete matrix ring D_k over a division ring D $(k>1)$ will be a homomorphic image of a subring of R and will satisfy (C). In particular if we choose $x=E$ and $y=E_{12}$, which gives a contradiction. Hence R is a field by the above discussion.

Lemma 5 Let R be a prime ring satisfying

$$(P_0) \qquad [x^m, y^n]^t-[x, y^s]\in Z(R)$$

for all $x, y\in R$, where m, n, s, and t are fixed positive integers such that $mt>1$, then

(1) R has no nonzero nilpotent elements.

(2) R has no zero divisors.

Proof (1) Suppose, to the contrary, that $a^2=0$ for some nonzero element a in R. By hypothesis we have

$$[a^m, (ax)^n]^t-[a, (ax)^s]\in Z(R) \text{ for all } x\in R. \text{ That is } (ax)^sa\in Z(R).$$

Then $(ax)^{s+2}=(ax)^saxax=xa(ax)^sax=0$, If $aR\neq 0$, then by Lemma 2 R has a nonzero nilpotent ideal, which is a contradiction since R is prime. Thus $aR=0$, which contradicts that $a\neq 0$.

(2) Let $a, b\in R$ such that $ab=0$, then $(bRa)^2=bRabRa=0$, and hence $bRa=0$.

R is prime, then $a=0$ or $b=0$.

Lemma 6 Let R be a semisimple ring satisfying (P_0), then R is commutative.

Proof The hypothesis is inherited by all subrings and all homomorphic images of R. Also, no complete matrix ring D_k over a division ring $D(k>1)$, satisfies the hypothesis, as a consideration of $x=E_{12}$ and $y=E_{11}$ shows. Using these facts and the structure theory of primitive rings, we may assume that R is a division ring.

If there exists $c\in Z(R)$ such that $c^{mt}\neq c$. Replacing x by cx in (P_0) we have $c^{mt}[x^m, y^n]^t - c[x, y^s]\in Z(R)$ for all $x, y\in R$. But $c^{mt}[x^m, y^n]^t - c^{mt}[x, y^s]\in Z(R)$, then $(c^{mt}-c)[x, y^s]\in Z(R)$. By Lemma 1 we have (1) $[x, y^s]\in Z(R)$. Now, replacing x by xy in (1) we get (2) $[x, y^s]y\in Z(R)$. Using (1) and (2) and again by Lemma 1 we obtain $y\in Z(R)$ unless $[x, y^s]=0$. If $y\in Z(R)$, then $[x, y^s]=0$. In either case, we have $xy^s=y^s x$ for all $x, y\in R$. By a result of Herstein [3], R is a field.

If $c^{mt}=c$ for all $c\in Z(R)$, then by Lemma 4 $Z(R)$ is a finite field. By [4, Theorem 1] R is finite-dimensional over $Z(R)$. Then R is a finite division ring, and hence R is a field.

Proof of Theorem 2 It suffices to show that a prime ring R satisfying (P) is commutative.

By Lemma 5 R has no zero divisors. Then the characteristic of R is 0 or a prime integer p. If R is of characteristic p, for $r\in R$ and $\bar{i}\in Z_p$, we define $\bar{i}r=ir$, where Z_p is a ring of integers modulo p, then R is an algebra over Z_p. If the characteristic of R is 0, localizing R at integers does not disturb our basic hypothesis (since $(m+n)t=s+1$), so that we may assume that R is an algebra over a field.

Pick $a, b\in R$, let S be the subalgebra of R generated by a and b, and $J(S)$ be the Jacobson radical of S. Then, by [1, Theorem 5] $J(S)=0$. Thus, by Lemma 6 S is commutative and therefore $ab=ba$. Hence R is commutative.

The following are immediate consequences of Theorem 2.

Corollary 1 Let R be a semiprime ring satisfying $[x^n, y]-[x, y^n]\in Z(R)$ for all $x, y\in R$ and a fixed integer $n>1$, then R is commutative.

Corollary 2 Let R be a semiprime ring. Then the following statements are equivalent:

(1) R is commutative.

(2) R satisfies $[x, y]^m-[x, y^{2m-1}]\in Z(R)$ for all $x, y\in R$ and a fixed integer $m>1$.

(3) R satisfies $[x,y]^n-[x^{2n-1}, y]\in Z(R)$ for all $x, y\in R$ and a fixed integer

$n>1$.

The ring of 3×3 strictly upper triangular matrices over some field provides an example to show that R is semiprime in Theorems 1 and 2 is not superfluous.

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半质环的两个交换性结果

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摘　　要

定理 1　设R是半质环，m，n是固定正整数，且$n>1$．如果R满足条件

$$(x^m y)^n - x^m y\in Z(R),\quad \forall x,\ y\in R,$$

则R是交换环．

定理 2　设R是半质环，m，n，s，t是固定正整数，且$(m+n)t=s+1$，$mt>1$．如果R满足条件

$$[x^m, y^n]^t - [x, y^s]\in Z(R),\quad \forall x,\ y\in R,$$

则R是交换环．