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p-Injectivity and Artinian Semisimple Rings*

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Abstract

Since several years, Artinian semisimple rings have drawn the attention of various authors (cf. for example, [1]—[10]). The purpose of this paper is to characterize Artinian semisimple rings in terms of P-injectivity. New characterizations of Artinian semisimple rings are obtained. Necessary and sufficient conditions for prime rings to be Artinian simple are given. Several interesting properties of P-injective rings are derived.

All rings in this paper are associative with unit, and all modules are unital. Let R be a ring. A left R-module M is called P-injective (cf. [8]) if, for any principal left ideal L, every left R-homomorphism $f: L\to M$ extends one of R to M, i.e., there exists $m\in M$ such that

$$f(r)=rm, \text{ for all } r\in L.$$

A ring R is called P-injective if, as a left R-module, it is P-injective.

Let N be a submodule of a left R-module M. N is called essential in M, if every nonzero submodule of M has nonzero intersection with N, in this case we also say that M is an essential extension of N. N is said to be a closed submodule of M, if N has no proper essential extensions inside M. If N is essential in M, then write $N\leq_e M$. For $x\in M$, we use $(N:x)$ to denote the left ideal $\{r\in R|rx\in N\}$ in R. For $a\in R$, we use $l(a)$ to denote the left ideal $\{r\in R|ra=0\}$ of R, and $l(a)$ is called a special left annihilator (cf. [11]). The singular submodule of a left R-module M is $Z(M)=\{y\in M|l(y)\leq_e R\}$, and M is called singular if $Z(M)=M$. Let Z, J denote respectivily the left singular ideal, the Jacobson radical of R, and $\mathrm{Soc}(Z), S$ denote respectivily the socle of right R-module Z and the right socle of R. A ring R is called left perfect, if it satisfies the descending chain condition on principal right ideals (cf. [12]). Let I be a subset of a ring R. I is called left T-nilpotent, if for every sequence $a_1, a_2, \cdots$ in I there is an n such that

$$a_1 \cdots a_n=0.$$

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It is clear that a left T-nilpotent left ideal I is nil.

Lemma 1[2] Let N be a submodule of a left R-module M. If $N\leq_e M$, then $(N:x)\leq_e R$ for all $x\in M$.

Proof For any $a\in R$ and $a\neq 0$, if $ax=0$, then $a\in(N:x)$, thus

$$0\neq Ra=(N:x)\cap Ra.$$

If $ax\neq 0$, then Rax is a nonzero submodule of M. Because $N\leq_e M$, thus $N\cap Rax\neq 0$, so that there exists $r\in R$ such that $rax\neq 0$ and $rax\in N$, whence

$$ra\neq 0 \text{ and } ra\in(N:x)\cap Ra.$$

At any rate, we always have $(N:x)\cap Ra\neq 0$, hence $(N:x)$ is essential in R.

Lemma 2[13] A ring R is P-injective if and only if $r(l(a))=aR$ for all $a\in R$.

Lemma 3[14] If a ring R is P-injective, then $Z=J$.

Lemma 4 Let R be a P-injective ring. Then R is left perfect if and only if R satisfies the ascending chain condition for special left annihilators.

Proof Assume that R is P-injective, and satisfies the ascending chain condition for special left annihilators. If $a_i\in R$, $i=1,2,\cdots$, and

$$a_1R\supseteq a_2R\supseteq\cdots\supseteq a_iR\supseteq\cdots$$

then

$$l(a_1)\subseteq l(a_2)\subseteq\cdots\subseteq l(a_i)\subseteq\cdots$$

Since R satisfies the ascending chain condition for special left annihilators, then there exists a natural number m such that

$$l(a_m)=l(a_{m+1})=\cdots$$

By Lemma 2, we have

$$a_mR=r(l(a_m))=r(l(a_{m+1}))=a_{m+1}R=\cdots$$

Therefore R is left perfect. Conversely, assume that R is P-injective and left perfect. If $a_i\in R$, $i=1,2,\cdots$, and

$$l(a_1)\subseteq l(a_2)\subseteq\cdots\subseteq l(a_i)\subseteq\cdots$$

then

$$r(l(a_1))\supseteq r(l(a_2))\supseteq\cdots\supseteq r(l(a_i))\supseteq\cdots$$

It follows from Lemma 2 that

$$a_1R\supseteq a_2R\supseteq\cdots\supseteq a_iR\supseteq\cdots$$

Since R is left perfect, then there exists a natural number n such that

$$a_nR=a_{n+1}R=\cdots$$

whence

$$l(a_n)=l(a_nR)=l(a_{n+1}R)=l(a_{n+1})=\cdots$$

i.e., R satisfies the ascending chain condition for left special annihilators.

Lemma 5 Let R be a P-injective ring. If R satisfies the ascending chain condition for left special annihilators then

(1) R/J is a semisimple Artinian ring;

(2) J is left T-nilpotent;

(3) Every nonzero right ideal of R contains a minimal right ideal.

The proof of this Lemma is straightforward (cf. Theorem 28.4, [15]).

Lemma 6 Let R be a P-injective ring. If R satisfies the ascendind chain condition for special left annihilators, then S is an essential right ideal in R.

Proof If S is not an essential right ideal in R, then there exists a nonzero right ideal K in R such that

$$S \oplus K \leq_e R.$$

By Lemma 5, K contains a minimal right ideal aR, thus

$$0 \neq aR \subseteq S \cap K$$

whence $S \cap K \neq 0$, a contradiction.

The following question is raised by Yue Chi Ming in [16]: Is a right P-injective ring with the maximal condition on right annihilators right Artinian? At present, we can not answer this question, but we weaken the maximal condition on right annihilators, and obtain the following interesting characterization of Artinian semisimple rings:

Theorem 1 The following conditions are equivalent:

(1) R is a semisimple Artinian ring;

(2) R is a P-injective ring, Z is a closed right ideal of R, and R satisfies the ascending chain condition for special left annihilators.

Proof Obviously, (1) implies (2).

Assume (2). It is clear that $\mathrm{Soc}(Z) \subseteq S$. Since S is semisimple right R-module, then there exists a right ideal K of R such that

$$\mathrm{Soc}(Z) \oplus K = S.$$

By Lemma 6, S is an essential right ideal in R, i.e.,

$$S \leq_e R \qquad (*)$$

If $Z \cap K \neq 0$, then it follows from Lemma 5(3) that

$$\mathrm{Soc}(Z) \cap (Z \cap K) \neq 0$$

which contradicts $\mathrm{Soc}(Z) \cap K = 0$. Therefore

$$Z \cap K = 0 \qquad (**)$$

It follows from (*) and (**) that

$$S = \mathrm{Soc}(Z) \oplus K \subseteq Z \oplus K \subseteq R$$

whence

$$Z \subseteq Z \oplus K \leq_e R.$$

Because Z is a closed right ideal in R, $(Z \oplus K)/Z$ is an essential submodule of the right R-module R/Z, i.e.,

$$(Z \oplus K)/Z \leq_e R/Z.$$

By Lemma 3 and Lemma 5 (1), we obtain

$$(Z\oplus K)/Z=R/Z.$$

Therefore there exists $k\in K$ such that

$$1-k\in Z=J$$

By Lemma 5 (2), J is T-nilpotent, thus there exists a natural number m such that $(1-k)^m=0$, it follows

$1=f(k)$, $f(k)$ is a polynomial of k with integer coefficients,

whence $1\in K$, $K=R$. It follows from (**) that

$$Z=0,\quad R=K=S.$$

This proves that R is Artinian semisimple.

In [7] it is shown that if every essential left ideal of a ring R is the annihilator, and R is left nonsingular, then R is Artinian semisimple. If we replace "nonsingularity" by "P-injectivity of singular simple R-modules", and weaken other the hypothesis, then we obtain the following characterization of Artinian semisimple rings:

Theorem 2 The following conditions are equivalent:

(1) R is a semisimple Artinian ring;

(2) Every maximal essential left ideal of R is the left annihilator and every singular simple left R-module is P-injective.

Proof Obviously, (1) implies (2).

Assume (2). For a left R-module N, let a left R-module M be an essential extension of N. If $N\neq M$, then there exists $x\in M-N$. By Lemma 1, we see

$$(N:x)\leq_e R,\quad (N:x)\neq R.$$

Thus there exists a maximal left ideal L of R such that

$$(N:x)\subseteq L\subseteq R,\quad L\leq_e R.$$

It follows from the hypothesis that

$$L=l(I),\ \text{for some subset } I \text{ of } R.$$

By maximality of L, there exists $a\in I$ such that

$$L=l(a),\quad a\neq 0.$$

If $r, r_1\in R$ such that $ra=r_1a$, then $r-r_1\in l(a)$, define

$$f: Ra\to R/l(a);\quad ra\mapsto r+l(a)$$

it is easily seen that f is a left R-homomorphism. Because $R/l(a)$ is a singular simple left R-module, there is $b\in R$ such that

$$1+l(a)=f(a)=ab+l(a)$$

which yields

$$1-ab\in l(a),\ a=aba$$

whence $e=ab\neq 1$ and $ee=e$. It follows that

$$L=l(e)=R(1-e),\ L\cap Re=0 \text{ and } Re\neq 0$$

which contradicts that $L\leq_e R$ and $L\neq R$. Therefore $N=M$, i.e., N has no proper essential extensions. By well-known theorems, N is a injective left R-module. It follows from Theorem 4.13 in [12] that R is Artinian semisimple.

Prime rings need not be Artinian simple. Conditions for prime rings to be Artinian simple are studied by various authors. In [2], it was shown that a prime Goldie ring R with nonzero socle is a simple Artinian ring. This motiuates the following theorem.

Theorem 3 Let R be a prime ring. If R satisfies the ascending chain condition for special left annihilators, $S\neq 0$, and S, as a left R-module, is P-injective, then R is a simple Artinian ring.

Proof (1) If a right ideal K of R satisfies $K\cap S=0$, then $KS\subseteq K\cap S=0$. Since R is a prime ring, therefore $K=0$, and S is an essential right ideal in R.

In the same way as above, we may prove that S is also an essential left ideal in R.

(2) Since R satisfies the ascending chain condition for special left annihilators, then the set

$$\{l(x)\mid x\in R \text{ and } x\neq 0\}$$

contains a maximal element $l(a)$. If $Z\neq 0$, then $aZa\neq 0$ because R is a prime ring. Thus there is $c\in Z$ such that $aca\neq 0$. It follows from the maximality of $l(a)$ that $l(a)=l(aca)$. Since $aca\in Z$, we have

$$l(a)=l(aca)\leq_e R$$

whence $l(a)\cap Rac\neq 0$. Thereby there exists $r\in R$ such that

$$rac\neq 0 \text{ and } raca=0$$

which implies $r\in l(aca)=l(a)$. Thus $ra=0, rac=0$ which contradicts $rac\neq 0$. Therefore $Z=0$.

(3) If $S\neq R$, then the set

$$\{l(x)\mid x\in R \text{ and } x\notin S\}$$

contains a maximal element $l(b)$ because R satisfies the ascending chain condition for special left annihilators. By (2), there exists a nonzero left ideal L of R such that

$$l(b)\oplus L\leq_e R.$$

By (1), we have $L\cap S\neq 0$, so that there is $d\in L\cap S$ and $d\neq 0$.

If $r, r_1\in R$ and $rdb=r_1db$, then $(r-r_1)db=0$ which implies that $(r-r_1)d\in l(b)\cap L=0$, $rd=r_1d$. Thereby we may define

$$f: Rdb\to S;\ rdb\mapsto rd$$

and it is easily seen that f is a left R-homomorphism. Since S is P-injective, then there is $h\in S$ such that

$$d = f(db) = dbh$$

which implies that $d(b-bhb)=0$, $d\in l(b-bhb)$. Since $b\overline{\in}S$ and $h\in S$, $b-bhb\overline{\in}S$. By maximality of $l(b)$, thus $l(b)=l(b-bhb)$. Thereby $d\in l(b-bhb)$ implies $db=0$, $d=dbh=0$ which contradicts $d\neq 0$. Therefore $S=R$, i.e., R is a semisimple Artinian ring.

(4) Let I be a nonzero ideal of R. By (3), there exists a left ideal T of R such that $I\oplus T=R$. This implies

$$IT\subseteq I\cap T=0$$

Since R is a prime ring, then $T=0$, $R=I$. This proves that R is a simple Artinian nian ring.

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p-内射性与Artin半单环

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摘 要

p-内射性在环论研究中有独特的作用，并且越来越被人们所重视．本文的目的是利用p-内射性来刻化Artin半单环，我们得到如下主要结果：

（1）环R是Artin半单的当且仅当R是p-内射的，R的左奇异理想是闭右理想，且R满足特殊左零化子升链条件；

（2）环R是Artin半单的当且仅当R的每个极大本质左理想是左零化子，并且任意奇异单左R-模是P-内射的；

（3）素环R是Artin单的当且仅当R的右基层$S\neq 0$是左p-内射的，并且R满足特殊左零化子升链条件．

这些结果不仅加深了对Artin半单环的认识，而且建立了半单环与某些特殊环之间的联系．

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广义拟内射模的自同态环

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摘 要

本文的目的，是推广［1］中定理1.22和［2］中命题1（1）．我们得到：

设R是环，且$Q=\mathrm{End}_R(M)$，其中M是广义拟内射模．那么有

（1） $J(Q)=Z(Q)$；

（2） $Q/J(Q)$是Von Neumann正则环．