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Endomorphism Rings of Generalized Quasi-Injective Modules*

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Abstract

The purpose of this paper is to generalize ([1], Theorem 1.22) and ([2], Proposition 1(1)). We obtain:

Let R be a ring, and let $Q=\mathrm{End}_R(M)$, where M is a generalized quasi-injective left R-module. Then

(1) $J(Q)=Z(Q)$;

(2) $Q/J(Q)$ is a Von Neumann regular ring.

All rings in this paper are associative with unit, and all modules are unital.

Let R be a ring. A submodule N of a left R-module M is said to be essential in M, if $N\cap K\neq 0$ whenever K is a non-zero submodule of M.If N is essential in M, then $N\leq_e M$. A submodule N of a left R-module M is said to be a complement submodule in M, if there exists a submobule K of Σ such that N is maximal with respect to the property $N\cap K=0$. A left R-module M is called generalizeb quasi-injective (cf. [2]) if, for any left submodule N which is isomorphic to a complement left submodule of M, every left R-homomorphism, from N into M may be extended to an endomorphism of M. It is clear that every quasi-injective left R-module is generalizeb quasi-injective (cf. [3]). A ring R is called a generalizeb quasi-injective ring if, as a left R-module, it is generalizeb quasi-injective (cf. [2]). A ring R is called Von Neumann regular (cf. [1]) if, for any $a\in R$ there exists $b\in R$ such that $a=aba$. As before, $J(R)$, $Z(R)$ will stand for the Jacobson radical and left singular ideal of a ring R respectively.

Lemma 1 (cf. [3]). Let M be a left R-module, and let $Q=\mathrm{End}_R(M)$. Then

$$Z(Q)=\{f\mid f\in Q \text{ and } \mathrm{Ker}(f)\leq_e M\}$$

is a two-sided ideal of Q.

Proof For any $f,g\in Z(Q)$, any $h\in Q$, we see

$$\mathrm{Ker}(f)\cap\mathrm{Ker}(g)\subseteq\mathrm{Ker}(f-g),$$

$$\mathrm{Ker}(f)\subseteq\mathrm{Ker}(hf),$$

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$$\mathrm{Ker}(fh)=h^{-1}(\mathrm{Ker}(f)),$$

whence

$$\mathrm{Ker}(f-g)\leq_e M,\qquad \mathrm{Ker}(hf)\leq_e M,\ \ \mathrm{Ker}(fh)\leq_e M.$$

Therefore

$$f-g,\ \ hf,\ \ fh\in Z(Q).$$

thus $Z(Q)$ is a two-sided ideal of Q.

Lemma 2 Let M be a generalized quasi-injective left R-module, and let $Q=\mathrm{End}_R(M)$. Then $Z(Q)\subseteq J(Q)$.

Proof For any $f\in Z(Q)$, it follows from Lemma 1 that

$$\mathrm{Ker}(gf)\leq_e M,\ \text{for all}\ g\in Q \qquad (*)$$

We see

$$\mathrm{Ker}(gf)\cap \mathrm{Ker}(1-gf)=0.$$

It follows from $(*)$ that $\mathrm{Ker}(1-gf)=0$, i.e., $1-gf$ is a monomorphism. Thus it is easily seen that $1-gf$ is a left R-isomorphism from M onto $(1-gf)M$. Assume h_1 is the inverse isomorphism of $1-gf$:

$$h_1:(1-gf)M\to M;\quad (1-gf)x\mapsto x$$

Since

$$M\cong(1-gf)M,\ \text{and}\ M\ \text{is complement in}\ M,$$

then by the hypothesis there exists $h\in Q$ such that h is an extension of h_1. We observe that

$$h(1-gf)x=h_1(1-gf)x=x,\ \text{for all}\ x\in M$$

whence

$$h(1-gf)=1,\ \text{for all}\ g\in Q.$$

It follows that Qf is left quasi-regular. Therefore $f\in J(Q)$, thus $Z(Q)\subseteq J(Q)$.

Theorem 3 Let M be a generalized quasi-injective left R-module. Then $Q/Z(Q)$ is a Von Neumann regular ring.

Proof For any $f\in Q$, there exists a complement submodule L of M such that $\mathrm{Ker}(f)\oplus L\leq_e M$. If x, $x'\in L$ with $fx=fx'$, then $x-x'\in\mathrm{Ker}(f)\cap L=0$, $x=x'$. Thus we may define

$$h: fL\to L;\quad fx\mapsto x$$

and it is easily seen that h is a left R-isomorphism. Because of M is a generalized quasi-injective left R-module, hence there is $g\in Q$ such that g is an extension of h. We obtain

$$g(fx)=h(fx)=x,\ \text{for all}\ x\in L \qquad (**)$$

For any $y\in\mathrm{Ker}(f)\oplus L$, we write

$$y=y_1+y_2,\quad y_1\in\mathrm{Ker}(f),\quad y_2\in L.$$

By $(**)$, $g(fy_2)=y_2$, and it follows from $y_1\in\mathrm{Ker}(f)$ that $fy_1=0$, thus we have

$$\begin{aligned}(f-fgf)(y) &= (f-fgf)(y_1)+(f-fgf)(y_2)\\ &= (fy_1-(fg)(fy_1))+fy_2-f(gf)(y_2)\\ &= fy_2-fy_2=0\end{aligned}$$

whence

$$\mathrm{Ker}(f)\oplus L\subseteq \mathrm{Ker}(f-fgf)\leq_e M.$$

Therefore $f-fgf\in Z(Q)$. This proves that $Q/Z(Q)$ is a Von Neumann regular ring.

Theorem 4 Let M be a generalized quasi-injective left R-module. Then $Z(Q)=J(Q)$.

Proof For any $f\in J(Q)$, it follows from Theorem 3 that there exists $g\in Q$ such that

$$f-fgf\in Z(Q) \qquad (***)$$

Since $J(Q)$ is a two-sided ideal of Q, thus $fg\in J(Q)$, so that there is $h\in Q$ such that $h(1-fg)=1$, whencethis

$$h(f-fgf)=(h(1-fg))f=1\cdot f=f.$$

By Lemma 1 and $(***)$, we obtain

$$f=h(f-fgf)\in Z(Q).$$

Therefore $J(Q)\subseteq Z(Q)$. By Lemma 2, we have $J(Q)=Z(Q)$.

As direct consequences, we have

Corollary 5 (cf. [1], Theorem 1.22). Let M be a quasiinjective left R-module, and let $Q=\mathrm{End}_R(M)$. Then

(a) $J(Q)=Z(Q)$;

(b) $Q/J(Q)$ is a Von Neumann regular ring.

Corollary 6 (cf. [2], Proposition 1) Let R be a generalized quasi-injective ring. Then

(a) $Z(R)=J(R)$;

(b) $R/J(R)$ is a Von Neumann regular ring.

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p-内射性与Artin半单环

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摘　　要

p-内射性在环论研究中有独特的作用，并且越来越被人们所重视．本文的目的是利用p-内射性来刻化Artin半单环，我们得到如下主要结果：

（1）环R是Artin半单的当且仅当R是p-内射的，R的左奇异理想是闭右理想，且R满足特殊左零化子升链条件；

（2）环R是Artin半单的当且仅当R的每个极大本质左理想是左零化子，并且任意奇异单左R-模是P-内射的；

（3）素环R是Artin单的当且仅当R的右基层$S\neq 0$是左p-内射的，并且R满足特殊左零化子升链条件．

这些结果不仅加深了对Artin半单环的认识，而且建立了半单环与某些特殊环之间的联系．

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广义拟内射模的自同态环

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摘　　要

本文的目的，是推广［1］中定理1.22和［2］中命题1（1）．我们得到：

设R是环，且$Q=\mathrm{End}_R(M)$，其中M是广义拟内射模．那么有

（1）$J(Q)=Z(Q)$；

（2）$Q/J(Q)$是Von Neumann正则环．