Journal of Mathematical Research and Exposition
Vol.11 No.4, Nov., 1991.

An Inquiry and Comment on Pearson's Formula*

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Abstact

For the Weibull distribution, Pearson's Formula is untenable at all. In fact, if $t=\frac{1}{m}\to 0.29073604950033318$, then the deviation Coefficient is given by $C_s(t)\approx 0.0405$, so that
$$P(t)=\frac{E\xi(t)-M_0(t)}{E\xi(t)-M_e(t)}\to\infty.$$

In [1 p.39], the authors remarked: "There is an interesting empirical relationship between the three quantities which appears to hold for unimodal Curves of moderate asymmetry, namely.

$$\text{mean}-\text{mode}=3(\text{mean}-\text{median}) \qquad (2\cdot 13)"$$

In all the works of Beyer [2], Armitage [3], Sachs [4], Mason [5], Freund [6], etc., and in a lot of other relevant bibliographies there are all the similar statements.

In order to find out the problem more clearly, let us take Pearson's Formula of the form:

$$\frac{E\xi-M_0}{E\xi-M_e}=3$$

where, $E\xi$ is mean, M_0 mode, M_e median.

Its meaning is: The distance between the arithmetic mean and the mode is three times larger than that from the former to the median, while the conditions of the formula being tenable can be concluded as: (1) the curve being unimodal; (2) it is an experimental formula, or approximated one, under the condition of slight skewness or moderate skewness.

In fact, under the above conditions as described, the formula can still be untenable.

Proposition. For the Weibull distribution, Pearson's Formula is untenable. More precisely, if $t=\frac{1}{m}\to 0.29073604950033318$, then $C_s(t)\to 0.0405164248$, and
$$\frac{E\xi(t)-M_0(t)}{E\xi(t)-M_e(t)}\to\infty.$$

* Received May, 12, 1990.

Proof. The probability density of the Weibull distribution is given by

$$P_w(x)=\begin{cases}\frac{m}{a}x^{m-1}e^{-\frac{x^m}{a}}, & x\geqslant 0\\ 0 & x<0\end{cases}$$

$$(m>0,\ a>0)$$

Note that from [7] p.800, $E\xi=a^{\frac{1}{m}}\Gamma(1+\frac{1}{m})$.

Also, from [8] p.841, its moment of order k is $u_k=a^{\frac{k}{m}}\Gamma(1+\frac{k}{m})$, so the derivation coefficient is given by

$$C_s=\frac{u_3-3u_1u_2+2v_1^3}{(\sqrt{v_2-v_1^2})^3}=\frac{\Gamma(1+\frac{3}{m})-3\Gamma(1+\frac{1}{m})\Gamma(1+\frac{2}{m})+2\Gamma^3(1+\frac{1}{m})}{(\sqrt{\Gamma(1+\frac{2}{m})-\Gamma^2(1+\frac{1}{m})})^3}$$

for $t=\frac{1}{m}$, and generally,

$$C_s(t)=\frac{\Gamma(1+3t)-3\Gamma(1+t)\Gamma(1+2t)+2\Gamma^3(1+t)}{(\sqrt{\Gamma(1+2t)-\Gamma^2(1+t)})^3} \qquad (1)$$

Next, $P'_w(x)=\frac{m^2}{a^2}x^{m-2}e^{-\frac{x^m}{a}}[\frac{a(m-1)}{m}-x^m]$, and if $x=\sqrt[m]{\frac{a(m-1)}{m}}=a^{\frac{1}{m}}(1-\frac{1}{m})^{\frac{1}{m}}$ $=a^t(1-t)^t$, then $P'_w(x)=0$. If $x<a^t(1-t)^t$, then $P'_w(x)>0$. If $x>a^t(1-t)^t$, then $P'_w(x)<0$.

And so the distribution is unimodal, and if $x=a^t(1-t)^t$, $P_w(x)$ gets the maximum value, hence $M_0=a^t(1-t)^t$. Let

$$M_0(t)=(1-t)^t. \qquad (2)$$

On the other hand, from

$\int_0^{M_e}\frac{m}{a}x^{m-1}e^{-\frac{x^m}{a}}dx=\frac{1}{2}$, namely $\int_0^{M_e}e^{-\frac{x^m}{a}}d(-\frac{x^m}{a})=-\frac{1}{2}$, one can find $M_e=a^{\frac{1}{m}}(\ln 2)^{\frac{1}{m}}=a^t(\ln 2)^t$. Let

$$M_e(t)=(\ln 2)^t. \qquad (3)$$

Since the function of arithmetic mean is

$$E\xi(t)=\Gamma(1+t). \qquad (4)$$

applying the formula at [9] p.257,

$\Gamma(1+t)=1+b_1t+b_2t^2+\cdots\cdots+b_7t^7+b_8t^8+\varepsilon(t)$

$|\varepsilon(t)|\leqslant 3\times 10^{-7}$ $\qquad 0\leqslant t\leqslant 1$

$b_1=-0.577191652$ $\quad b_2=0.988205891$ $\quad b_3=-0.897056937$ $\quad b_4=0.918206857$

$b_5=-0.756704078$ $\quad b_6=0.482199394$ $\quad b_7=-0.193527818$ $\quad b_8=0.035868343$

If $t=0.2907360495003318$, we can obtain $E\xi(t)=\Gamma(1+t)=M_e(t)=(\ln 2)^t=0.8989224429$, and then $M_0(t)=0.9049497708$, $C_s(t)=0.0405164248$.

Obviously, $E\xi(t)$, $M_e(t)$, $M_0(t)$ are continuous functions of t on [0,1], so that

$$\lim_{t\to 0.2907360495003318}\frac{E\xi-M_0}{E\xi-M_e}=\lim_{t\to 0.2907360495003318}\frac{E\xi(t)-M_0(t)}{E\xi(t)-M_e(t)}=\infty$$

Then, the value of $C_s(t)$ compared with $C_s(10)\approx 69900$, $c_s(20)\approx 1.129\times 10^{10}$ is of very ting skewness.

We can see from the following table that, under the condition $C_s\approx 0.0405$, the value of $P(t)=\dfrac{E\xi(t)-M_0(t)}{E\xi(t)-M_e(t)}$ can be unlimitedly increased and is not equal to 3 at all. Hence Pearson's Formula is untenable.

t	$P(t)$	$C_s(t)$
0.28	6.5	0.0074489161
0.2906	267	0.0400998927
0.290735	34212	0.0405132119
0.2907360045	7977716	0.0405164111
0.290736 0494	357835319	0.0405164245
0.29073604950 03316	54289342820509	0.0405164248
0.29073604950 03318	∞	0.0405164248
0.29073604950 03320	−54289342820507	0.0405164248
0.2907360495 1	−3713615348	0.0405164249
0.2907360496	−330216718	0.0405164251
0.29073605	−71852450	0.0405164264

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