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On Finite Element Algorithm of Transient Heat Conduction Equation*

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To eliminate oscillation and overbounding in numerical solutions of transient heat conduction equation, the authors have proposed several criteria in[1]. For 1-D passive heat conduction with constant parameters and boundary conditions, lower-and upper bound formulas for Δt can be derived. Computation indicates that the lower bound obtained in[1]can not be released while the upper bound can be quite higher than that in[1].The authors will prove that there exist higher upper bounds and give a highcr upper bound formula for Δt.

Consider the following heat conduction equation

$$\frac{\partial T}{\partial t}=a\nabla^2 T+f(\underset{\sim}{x},t),T=T_0,\underset{\sim}{x}\in V;$$

$$T=T_1,\underset{\sim}{x}\in S_1;\ \frac{\partial T}{\partial n}=\rho,\underset{\sim}{x}\in S_2;\ \frac{\partial T}{\partial n}=\beta(T_3-T),\underset{\sim}{x}\in S_3. \qquad (1)$$

Its finite element algorithm is

$$[A]\underset{\sim}{q}^{n+1}=[B]\underset{\sim}{q}^n+\Delta t(\theta\underset{\sim}{F}^{n+1}+(1-\theta)\underset{\sim}{F}^n);\ [A]=[M]+\theta\Delta t[K];$$
$$[B]=[M]+(\theta-1)\Delta t[K];\ [M]=\sum_i\iiint_{V_i}\underset{\sim}{N}^T(\underset{\sim}{x})\underset{\sim}{N}(\underset{\sim}{x})dv;$$
$$[K]=\sum_i\iiint_{V_i}a\nabla\underset{\sim}{N}^T(\underset{\sim}{x})\nabla\underset{\sim}{N}(\underset{\sim}{x})dv+\sum_j\iint_{S_j}a\beta\underset{\sim}{N}^T(\underset{\sim}{x})\underset{\sim}{N}(\underset{\sim}{x})ds;$$
$$F=\sum_i\iiint_{V_i}f\underset{\sim}{N}^T(\underset{\sim}{x})dv+\sum_j\iint_{S_j}a\beta T_3\underset{\sim}{N}^T(\underset{\sim}{x})ds. \qquad (2)$$

Here q^n is temperature vector at time step t_n, $\underset{\sim}{N}(\underset{\sim}{x})$ is element shape function matrix, $0.5\leqslant\theta\leqslant 1$.Other symbols are defined in[1].

One general criterion is

$$[A]^{-1}[B]\geqslant 0,\ [A]^{-1}\geqslant 0 \qquad (3)$$

Under certain conditions, a lower-and upper bound formula can be derived from Eq.(3).

$$-\min_{\substack{i,j\\ i\neq j}}(M_{ij}/(\theta K_{ij}))\leqslant\Delta t\leqslant\min_i(M_{ii}/((1-\theta)K_{ii})). \qquad (4)$$

The authors are to prove that there exist higher upper bounds for Δt than that given in Eq.(4). Note that Eq.(4) stems from Eq.(3) under certain conditions. Accordingly, the proof should be derived from Eq.(3) either.

Rearrange the first formula of Eq.(4).

$$[A]^{-1}[B]=[I]-[A]^{-1}[K]\Delta t=[A]^{-1}[M]+(\theta-1)\Delta t\,[A]^{-1}[K]\geqslant 0, \qquad (5)$$

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thus

$$\Delta t\,[A]^{-1}[K]=([I]-[A]^{-1}[M])/\theta. \tag{6}$$

Substitute Eq.(6) into Eq.(5) and yield

$$[A]^{-1}[M]\geqslant(1-\theta)[I]. \tag{7}$$

Since $[A]^{-1}>0$, $[M]>0$, then $([A]^{-1}[M])_{ii}>(1-\theta)$ is enough for Eq.(7) to hold. Resolve positive-definite $[A]$ into

$$[A]=[D]^{1/2}([I]-[P])[D]^{1/2}, \tag{8}$$

where $D_{ii}=A_{ii}$, $D_{ij}=0$, $P_{ij}=-A_{ij}/\sqrt{D_{ii}D_{jj}}\geqslant 0$.

$$[A]^{-1}=[D]^{-1/2}([I]-[P])^{-1}[D]^{1/2}=[D]^{-1/2}([I]+[P]+[P]^2+\cdots)[D]^{-1/2} \tag{9}$$

Since the maximum eigenvalue of $[P]$ is less than one

$$([A]^{-1}[M])_{ii}=([M]+[P][M]+[P]^2[M]+\cdots)_{ii}/D_{ii}\geqslant(1-\theta). \tag{10}$$

If taking only the first two terms of the infinite series in Eq.(10), the authors derive

$$\Delta t\leqslant\min_i\,(M_{ii}/(K_{ii}(1-\theta))+\sum_{j\neq i}(-A_{ij}M_{ji}/\sqrt{A_{ii}A_{jj}})/(K_{ii}(1-\theta))), \tag{11}$$

which shows that the upper bound can be greater than that in Eq.(4). In fact, if taking only the first term, the upper bound given in Eq.(4) is derived.

Consider a 1-D heat conduction problem: the left boundary is adiabatic, the right boundary is convective; β denotes convective factor. Using linear element with equal length Δx, an explicit upper bound is derived

$$r\leqslant 1/(3(1-\theta)(1+\beta\Delta x))+\sqrt{2}\,r\,(r-1/6\theta)/(12\sqrt{(1+r/3)(1+\beta\Delta x+r/3)})' \tag{12}$$

where $r=a\Delta t/\Delta x^2$. Since $r\geqslant 1/6\theta$ (the lower bound in this case[1]), this upper bound is greater than the explicit upper bound of the same case given in[1].

Since an infinite series is involved in Eq.(10) which is identical with Eq.(3), upper bounds that are still greater exist. Let $\Delta t\to\infty$, than $[A]\to\infty$, that is to say $[A]^{-1}\to 0$, Eq.(7) can not hold. This mean that although the upper bound can be much greater, it approaches a finite limit and can not be an infinity.

Referenec

[1] Ouyang Huajiang, Xiao Ding, Applied Mathematics and Mechanics (English Edition), 1989, Vol.10, No.12, 1179~1185.