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A Short Proof on the Bandwidth of A Graph and Its Complement*

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A labeling of a graph $G=(V,E)$ is a bijection $f: V\to\{1,2,\cdots,|V|\}$. For a labeling f, denote

$$B(G, f)=\max_{xy\in E}|f(x)-f(y)|.$$

The bandwidth of G is defined by

$$B(G)=\min_f B(G, f)$$

P Z. Chinn, F. R. K. Chung, P. Erdös and R. L. Graham[1] proved that

$$B(G)+B(\bar{G})\geq n-2$$

where $\bar{G}$ is the complement of G. In this note we present a short proof.

We will use a variant of Harper's lower bound[2] as follows.

Remark 1 For any labeling f,

$$B(G, f)\geq \max_{1\leq m\leq n}|N(f^{-1}([1, m]))|$$

where $[a, b]\triangleq\{x\in Z|a\leq x\leq b\}$, $N(S)\triangleq\{y\in V\setminus S|\ \exists x\in S$ such that $xy\in E(G)\}$ $(S\subseteq V)$.

Two special graphs P_n^k and P_n^k will be taken into account, where

$$V(P_n^k)=V(\bar{P}_n^k)=\{1, 2, \cdots, n\},$$
$$E(P_n^k)=\{ij|0<j-i\leq k\},$$
$$E(\bar{P}_n^k)=\{ij|k+1\leq j-i\leq n-1\}.$$

When drawing these graphs, we always put the vertices $1, 2, \cdots, n$ successively on a line (as shown in Fig 1).

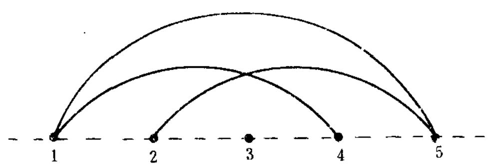

Figure 1 ($\bar{P}_5^2$)

Remark 2 $B(\bar{P}_n^k)\geq n-k-2$.

Proof For any given labeling f, denote $u_i=f^{-1}(i)$, $i=1,2,\cdots,n$. In the sequence $(u_1, u_2, \cdots, u_n)$, let u_m be the first vertex which is adjacent to a previous vertex, say u_l $(l<m)$. Without loss of generality, we may assume that

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$u_l < u_m$. Then $k+1 \leq u_m - u_l \leq n-1$.

We denote $S = \{u_1, u_2, \cdots, u_m\} = f^{-1}([1, m])$ and $u^* = \min\{u_1, u_2, \cdots, u_m\} \leq u_l$. Since $S \setminus \{u_m\}$ is an independent set, it follows that

$$S \subseteq [u^*, u^* + k] \cup \{u_m\}.$$

Moreover, each vertex $x < u^*$ is adjacent to u_m; each vertex $y > u^* + k$ is adjacent to u^*. Thus

$$N(S) \supseteq V \setminus ([u^*, u^* + k] \cup \{u_m\}).$$

Therefore

$$|N(S)| \geq n - k - 2.$$

By Remark 1,

$$B(G, f) \geq |N(S)| \geq n - k - 2$$

for any labeling f. This completes the proof. ∎

Remark 3 $B(G) + B(\overline{G}) \geq n-2$.

Proof Suppose $B(G) = k$. Then $G \subseteq P_n^k$, $\overline{G} \supseteq \overline{P}_n^k$. Thus $B(\overline{G}) \geq B(\overline{P}_n^k) \geq n-k-2$, and so $B(\overline{G}) + B(G) \geq n-2$. ∎

It is easy to construct a labeling which achieves the bound in Remark 2. So $B(\overline{P}_n^k) = n-k-2$. By using the same method, we can obtain that $B(\overline{C}_n^k) = \min\{n-k-2, 2(n-2k-2)\}$.

References

[1] P. Z. Chinn, F. R. K. Chung, P. Erdös and R. L. Graham, On the bandwidth of a graph and its complement. The Theory and Applications of Graphs (G. Chartrand, Ed.), Wiley (1981), 243—253.

[2] L. H. Harper, Optimal numberings and isoperimetric problem on graphs. J. Combin. Theory 1 (1966), 385—393.

关于图与补图的带宽的一个简单证明

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关于图与补图的带宽，P. Z. Chinn, F. R. K. Chung, P. Erdös 和 R. L. Graham 证明了 $B(G) + B(\overline{G}) \geq n-2$. 本文给出一个简单证明.