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On Weakly Commutative po-Semigroups *

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Abstract We introduce the concept of weakly commutative po-semigroups. We study the semilattice decompositions of this type of semigroups into their archimedean subsemigroups and we give a characterization of such semigroups analogous to the corresponding result on semigroups without order. As a consequence, the characterization of po-semigroups and the corresponding characterization on semigroups without order can be obtained.

Keywords po-semigroups, poe-semigroups, weakly commutative po- semigourps, archimedcan subsemigroups

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A po-semigroup(: ordered semigroup) is an ordered set S at the same time a semigroup such that for any $a, b, x \in S$

$$a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx.$$

A subset $I(\neq \emptyset)$ of S is called an ideal of S if

1) $SI \subseteq I$ and $IS \subseteq I$;
2) $a \in I, b \in S, b \leq a \Rightarrow b \in I$.

An ideal I of S is called prime if for any $a, b \in S, ab \in I \Rightarrow a \in I$ or $b \in I$.

A subsemigroup F of a po-semigroup S is called a filter of S if

1) $a, b \in S, ab \in F \Rightarrow a \in F$ and $b \in F$;
2) $a \in F, b \in S, b \geq a \Rightarrow b \in F$.

We denote by $N(x)$ the filter of S generated by $x (x \in S)$. Let $\mathcal{N}$ be an equivalence relation defined by

$$\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}.$$

Definition 1 *A po-semigroup S is called weakly commutative if for every $x, y \in S$ there exists a natural number n such that $(xy)^n \in (ySx]$.*

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Equivalent Definition

$$(xy)^n \leq yax \text{ for some } a \in S.$$

Definition 2 *Let S be a po-semigroup and T be a subsemigroup of S, T will be called archimedean if for every $a, b \in T$ there exists a natural number m such that $a^m \in (TbT]$.*

Equivalent definition

$$a^m \leq xby \text{ for some } x, y \in T.$$

We refer to [1] and [2] for all undefined terms and symbols in algebraic theory and ordered theory of semigroups respectively.

Lemma 1 *Let S be a weakly commutative po-semigroup. Then for any $x, y, b \in S$ and positive integern there exists some positive integer m, k, l such that $(xby)^m \in (Sb^nS], (xby)^k \in (Sb^n]$ and $(xby)^l \in (b^nS]$.*

Proof It is trivial for $n = 1$. Now suppose for positive integer n, there exists m such that $(xby)^m \in (Sb^nS]$, i.e., $(xby)^m \leq ub^nv$ for some $u, v \in S$. Since S is weakly commutative, so there exists some positive integer k such that $(ub^nv)^k \in (vSub^n) \subseteq (Sb^n]$, i.e., $(ub^nv)^k \leq wb^n$ for some $w \in S$ and also there exists 1 such that $(xby)^l \leq bz$ for some $z \in S$. Therefore we have $(xby)^{mk+1} \leq (ub^nv)^k(xby)^l \leq wb^nbz = wb^{n+1}z$, i.e., $(xby)^{mk+1} \in (Sb^{n+1}S]$.

Similarly we can prove the rest parts of lemma 1.

Lemma 2 *Let S be a weakly commutative po-semigroup. Then for any $a, b, \in S, a \leq b$ there exists m, n, i, j, k, l such that $a^m \in (abSab]$ and $(ab)^n \in (aSa], a^i \in (baSba]$ and $(ba)^j \in (aSa]$ and $a^k \in (babSbab]$ and $(bab)^l \in (aSa]$.*

Proof Clearly $a^5 = aaaaa \leq abaab$, which implies $a^5 \in (abSab]$. Since S is weakly commutative, hence there exists some n such that $(ab)^{n-1} \in (bSa]$, which implies $(ab)^{n-1} \leq xa$ for some $x \in S$. Thus we have $(ab)^n = (ab)(ab)^{n-1} \leq abxa$, so $(ab)^n \in (aSa]$.

Similarly we can prove the rest parts of lemma 2.

Theorem 1 *Let S be a po-semigroup. Then S is weakly commutative if and only if $N(x) = \{y \in S | x^n \in (ySy] \text{ for some } n\}$ for every $x \in S$.*

Proof Necessity. For $x \in S$, let

$$T = \{y \in S | x^n \in (Sy] \text{ for some } n\}.$$

We show first that T is a filter of S. Suppose $y, z \in T$, then $x^m \in (Sy]$ and $x^n \in (Sz]$ for some m, n, hence there exist $a, b \in S$ such that $x^m \leq ay$ and $x^n \leq bz$. Since S is weakly commutative, we have $(bz)^r \in (zSb]$ for some r, i.e., $(bz)^r \leq zc$ for some $c \in S$. Consequently

$$x^{m+nr} \leq (ay)(bz)^r \leq (ay)(zc) = (ayz)c.$$

Again by weakly commutativity, we have $((ayz)c)^k \leq d(ayz)$ for some $d \in S$ and for some k. Hence

$$x^{(m+nr)k} \leq d(ayz) = (da)(yz),$$

i.e., $x^{(m+nr)k} \in (Syz]$ and thus $yz \in T$. Now suppose that $yz \in T$. Hence $x^m \in (Syz] \subseteq (Sz]$ for some m so that $z \in T$. Since S is weakly commutative, we have $(a(yz))^n = ((ay)z)^n \in (zSay] \subseteq (Sy]$ for some n and so $x^{mn} \in (Sy]$, hence $y \in T$. Again suppose that $y \in T, z \in S, z \geq y$. Then $x^m \in (Sy]$ for some m, i.e., $x^m \leq ay$ for some $a \in S$ and for some m. By $y \leq z$, we have $x^m \leq az$, i.e., $x^m \in (Sz]$ and hence $z \in T$. Therefore T is a filter of S. Since $x \in T$, by the minimality of $N(x)$, we must have $N(x) \subseteq T$. On the other hand, for every $y \in T$, we have $x^n \leq ay$ for some $a \in S$ and for some n. Since $x^n \in N(x)$, by $x^n \leq ay$, we have $ay \in N(x)$ and thus $y \in N(x)$, i.e., $T \subseteq N(x)$ so that $T = N(x)$.

By symmetry we can prove also that

$$N(x) = \{y \in S | x^m \in (yS] \text{ for some } m\},$$

which implies

$$\begin{aligned} N(x) &= \{y \in S | x^n \in (Sy] \text{ for some } n\} = \{y \in S | x^m \in (yS] \text{ for some } m\} \\ &= \{y \in S | x^k \in (ySy] \text{ for some } k\}. \end{aligned}$$

Sufficiency.Let $x, y \in S$. Since $\mathcal{N}$ is a semilattice congruence on S, hence we have $yx \in N(xy)$, which implies $(xy)^n \in (yxSyx) \subseteq (ySx]$ for some n. Thus S is weakly commutative.

Theorem 2 *Let S be a weakly commutative po-semigroup. Then the following are true:*

i) S is (not uniquely in general (c.f. [4])) the semilattice of archimedean subsemigroups.

ii) $\mathcal{N}$ is the greateast semilattice congruence on S such that congruence class N_x is archimedean for any $x \in S$.

iii) $\mathcal{N} = \{(x,y) \in S \times S | x^m \in (SyS]$ and $y^n \in (S \times S]$ for some $m, n\}$.

Proof i) It suffices to show that each $\mathcal{N}$-class N_x is an archimedean subsemigroup. For any $a, b \in N_x$, then $N(a) = N(x) = N(b)$, which implies $b \in N(a)$. Hence there exists some positive integer n such that $a^n \in (bSb]$, which implies $a^n \leq byb$ for some $y \in S$. Since $N(x)$ is a filter, hence $byb \in N(a) = N(b)$ so that $by \in N(b)$. Again, $b \in N(by)$ and $b \in N(byb)$, which implies $N(by) = N(byb) = N(b) = N(x)$. Therefore, we have $by, byb \in N_x$. It follows that from $a^{2n} \leq (by)b(byb) N_x$ is an archimedean subsemigroup of S.

ii) Let σ be a semilattice congruence on S such that for every $x \in S$ congruence class σ_x is archimedean. Let $(a,b) \in \sigma$. Since $a \in \sigma_b$ and σ_b archimedean, there exists n such that $a^n \leq ubv$ for some $u, v \in \sigma_b$. Then $ubv \in N(a)$ and $b \in N(a)$ thus $N(b) \subseteq N(a)$. From $b \in \sigma_a$, by symmetry, we have $N(a) \subseteq N(b)$. Hence $(a,b) \in \mathcal{N}$.

iii) Let $M(x) = \{y \in S | x^n (SyS]$ for some $n\}$ and let

$$\mathcal{M} = \{(x,y) | x^m \in (SyS] \text{ and } y^n \in (SxS] \text{ for some } m, n\}.$$

Clearly $\mathcal{M} = \{(x,y) | M(x) = M(y)\}$. Since S is weakly commutative, thus $N(x) = \{y \in S | x^n \in (ySy]$ for some $n\}$. Let $y \in M(x)$. Then $x^n \in (SyS]$ for some n, which implies $x^n \leq ayb$ for some $a, b \in S$ and also there exists m, k such that $(ayb)^m \leq yu$

and $(ayb)^k \leq vy$ for some $u, v \in S$ so that $x^{n(m+k)} \leq (ayb)^{m+k} = (ayb)^m(ayb)^k \leq yuvy$, i.e., $y \in N(x)$. Hence $M(x) \subseteq N(x)$. Let $y \in N(x)$. Then we have $x^n \leq yay$ for some $a \in S$ and for some n, so that $x^{2n} \leq (yay)(yay) = (ya)y(yay)$, i.e., $y \in M(x)$. Thus $M(x) \supseteq N(x)$ so that $N(x) = M(x)$. Therefore $\mathcal{N} = \mathcal{M}$.

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References

[1] A.H.Clifford and G.B.Preston, *The algebraic theory of semigroups*, Amer. Soc. Math. Surveys 7, Providence, Vol.I, 1964.

[2] N.Kehayopulu, *Remark on ordered semigroups*, Math. Japonica, 35, No.6(1990), 1061-1063.

[3] N.Kehayopulu, *On weakly commutative poe-semigroups*, Semigroup Forum, 34(1987), 367-370.

[4] N.Kehayopulu, P.Kiriakuli, S.Hanumantha Rao and P.Lakshmi, *On weakly commutative poe-semigroups*, Semigroup Forum, 41(1990), 373- 376.

[5] P.V.Ramana Murty and S.Hanumantha Rao, *On a problem in poe-semigroups*, Semigroup Forum, 43(1991), 260-262.

关于弱交换po-半群

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摘　要

在本文中我们引入弱交换po-半群的概念，并研究这类半群到其Archimedes子半群的半格分解，给出了这类半群类似于无序半群相应结果的一个刻画．作为推论，我们得到弱交换poe-群和无序半群的相应刻画．