基于 $xL_{n-1}^{(\alpha)}(x)$ 之零点的 $(0,1,\cdots,m-2,m)$ 插值

史应光
(中国科学院计算中心，北京 100080)

摘 要

本文给出了基于 $xL_{n-1}^{(\alpha)}(x)$ 之零点的 $(0,1,\cdots,m-2,m)$ 插值的正则性的充要条件，其中 $L_{n-1}^{(\alpha)}(x)$ 为 $(n-1)$ 次 Laguerre 多项式. 同时基函数(若存在的话) 的明显表达式也在文中给出. 再者，还证明了，若该插值问题有无穷多个解，则其解的一般形式为 $f_0(x)+Cf_1(x)$，这里 C 为任意常数.

Journal of Mathematical Research & Exposition
Vol.15, No.3, 319–328, Aug. 1995

$(0,1,\cdots,m-2,m)$ Interpolation on the Zeros of $xL_{n-1}^{(\alpha)}(x)$ *

Shi Yingguang
(Computing Center, Academia Sinica, Beijing 100080)

Abstract A necessary and sufficient condition for regularity of $(0,1,\cdots,m-2,m)$ interpolation on the zeros of $xL_{n-1}^{(\alpha)}(x)(\alpha>-1)$ in a managealbe form is established, where $L_{n-1}^{(\alpha)}(x)$ is the $(n-1)$- th Laguerre polynomial. Meanwhile, the explicit representation of the fundamental polynomials, when they exist, is given. Moreover, we show that if the problem of $(0,1,\cdots,m-2,m)$ interpolation has an infinity of solutions then the general form of the solutions is $f_0(x)+Cf_1(x)$ with an arbitrary constant C.

Keywords Birkhoff interpolation, Laguerre Polynomial, regularity.

Classification AMS(1991) 41A05/CCL O174.41

1. Introduction

Let us consider a system A of nodes

$$0\le x_1<x_2<\cdots<x_n,\quad n\ge 2. \tag{1.1}$$

Let $\mathbf{P}_n$ be the set of polynomials of degree at most n and $m\ge 2$ a fixed integer. The problem of $(0,1,\cdots,m-2,m)$ interpolation is, given a set of numbers

$$y_{kj},\ k\in N:=\{1,2,\cdots,n\},\ j\in M:=\{0,1,\cdots,m-2,m\} \tag{1.2}$$

to determine a polynomial $R_{mn-1}\in\mathbf{P}_{mn-1}$ (if any) such that

$$R_{mn-1}^{(j)}(x_k)=y_{kj},\quad \forall k\in N,\forall j\in M. \tag{1.3}$$

If for an arbitrary set of numbers y_{kj} there exists a unique polynomial $R_{mn-1}\in\mathbf{P}_{mn-1}$ satisfying (1.3) then we say that the problem of $(0,1,\cdots,m-2,m)$ interpolation on A is regular (otherwise, singular) and $R_{mn-1}(x)$ can be uniquely written as

$$R_{mn-1}(x)=\sum_{\substack{k\in N\\ j\in M}} y_{kj}r_{kj}(x), \tag{1.4}$$

*Received July 25, 1993, Project Supported by National Natural Science Foundation of China.

where $r_{kj} \in \mathbf{P}_{mn-1}$ satisfy

$$r_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k,\nu \in N, \quad j,\mu \in M \tag{1.5}$$

and are called the fundamental polynomials. In particular, for convenience of use we set

$$\rho_k(x) := r_{km}(x), \quad k = 1, 2, \ , n. \tag{1.6}$$

In [1] and [2] the exact condition of regularity on the parameter $\alpha \geq -1$ of the Laguerre polynomials $L_n^{(\alpha)}(x)$ is found for $(0, 2)$ interpolation based on the zeros of these polynomials, while the problem of determining the fundamental polynomials is partially solved for $\alpha = -1$. For $\alpha > -1$ the representation of fundamental polynomials is given only in the case when α is an odd integer and only on $(-\infty, 0)$, while a representation on $[0, \infty)$ would be more important. Follwing the main idea of [1] and [2] we give here a necessary and sufficient condition of regularity of $(0, 1, \cdots, m-2, m)$ interpolation on the zeros of $xL_{n-1}^{(\alpha)}(x)(\alpha > -1)$. Meanwhile we develop a method of finding the explicit representation of the fundamental polynomials when they exist without exception. Finally, when the problem of $(0, 1, \cdots, m-2, m)$ interpolation on A is not regular then for a given set of numbers y_{kj} either there is no polynomial $R_{mn-1}(x)$ satisfying (1.3) or there is an infinity of polynomials with the property (1.3). Moreover, we show that in the case of infinity many solutions the general form of the solutions is

$$R_{mn-1}(x) = f_0(x) + Cf_1(x),$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials and C is an arbitrary number.

2. An Auxiliary Lemma

We first state a lemma given by the author in [3]. To this end we introduce the fundamental polynomials of $(0, 1, \cdots, m-1)$ interpolation. Let $A_{kj}, B_k \in \mathbf{P}_{mn-1}$ be defined by

$$A_{kj}^{(\mu)}(x_\nu) = \delta_{k\nu}\delta_{j\mu}, \quad k,\nu = 1, 2, \cdots, n, \quad j,\mu = 0, 1, \cdots, m-1 \tag{2.1}$$

and

$$B_k(x) := A_{k,m-1}(x) = \frac{1}{m!}(x - x_k)^{m-1} l_k^m(x), \quad k = 1, 2, \ , n, \tag{2.2}$$

where

$$l_k(x) := \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)}, \ \omega_n(x) = c(x - x_1)(x - x_2)\cdots(x - x_n)(c \neq 0). \tag{2.3}$$

Then we have

Lemma *If there is an index $i, 1 \leq i \leq n$, such that $\rho_i \in \mathbf{P}_{mn-1}$ with the properties (1.5) exists uniquely, then the problem of $(0, 1, \cdots, m-2, m)$ interpolation is regular and*

$$r_{kj}(x) = A_{kj}(x) - \sum_{\nu=1}^{n} A_{kj}^{(m)}(x_\nu)\rho_\nu(x), \quad k = 1, 2, \cdots, n, \quad j = 0, 1, \cdots, m-2. \tag{2.4}$$

Remark Since the explicit representation for A'_{kj}s is well known, by (2.4) it is sufficient to find the one for ρ'_ks.

3. Main Results

In what follows let n be fixed and (1.1) the zeros of $xL_{n-1}^{(\alpha)}(x)(\alpha > -1)$. Of course, $x_1 = 0$ and $x_2, \cdots, x_n$ are the zeros of $L_{n-1}^{(\alpha)}(x)$. Write

$$\gamma := \frac{1}{2}(m-1)(\alpha-1), \quad \gamma := (m\gamma - m + 1)n - (m+1)\gamma, \tag{3.1}$$

$$\gamma_k := \frac{(-1)^k \begin{pmatrix} n-1+\alpha \\ n-1-k \end{pmatrix}}{k!}, \quad k = 0, 1, \cdots, n-1. \tag{3.2}$$

It is well known that [2]

$$L_{n-1}^{(\alpha)}(x) = \sum_{k=0}^{n-1} \gamma_k x^k \tag{3.3}$$

satisfies the differential equation

$$xL_{n-1}''^{(\alpha)}(x) + (\alpha + 1 - x)L_{n-1}'^{(\alpha)}(x) + (n-1)L_{n-1}^{(\alpha)}(x) = 0. \tag{3.4}$$

Theorem 1 *The problem of $(0, 1, \cdots, m-2, m)$ interpolation on the zeros of $\omega_n(x) := xL_{n-1}^{(\alpha)}(x)(\alpha > -1)$ is regular if and only if*

$$D_n(\alpha) = 2\gamma_0^2(n-1)(\frac{m-1}{2})^n + \gamma_0 \sum_{k=1}^{n-1} \frac{(-1)^k}{k}(\frac{m-1}{2})^{n-1-k}$$

$$\begin{pmatrix} n-1+\alpha \\ n-1-k \end{pmatrix} \begin{pmatrix} \gamma-2 \\ k-1 \end{pmatrix} \left\{ r - \frac{\gamma(\alpha+1)(n-1-k)}{(k+1)(k+1+\alpha)} \right\} \neq 0. \tag{3.5}$$

If the problem is regular, then for each $i, 1 \leq i \leq n$, the fundamental polynomial $\rho_i(x) := \rho_i(x; \alpha)$ is given by

$$\rho_i(x) = \omega_n^{m-1}(x) q_i(x), \tag{3.6}$$

in which $q_i \in \mathbf{P}_{n-1}$ is of the form

$$q_i(x) = x^\gamma e^{-\frac{(m-1)x}{2}} \left\{ d_i + \int_1^x [Q_i(t) - (c_i + e_i t) L_{n-1}^{(\alpha)}(t)] t^{-\gamma-1} e^{\frac{(m-1)t}{2}} dt \right\}, \tag{3.7}$$

with certain constants d_i, c_i and e_i, where

$$Q_i(x) = \frac{x_i l_i(x)}{m! [\omega_n'(x_i)]^{m-1}} \tag{3.8}$$

Proof By the definition of ρ_i we may set

$$\rho_i(x) = \omega_n^{m-1}(x)q_i(x), \tag{3.9}$$

where $q_i \in \mathbf{P}_{n-1}$ will be determined later. Then the requirement (1.5) yields

$$[\omega_n^{m-1}(x)q_i(x)]_{x=x_k}^{(m)} = \delta_{ik}, \quad k = 1, 2, \cdots, n. \tag{3.10}$$

It is easy to see that

$$[\omega_n^{m-1}(x)]_{x=x_k}^{(m)} = \frac{1}{2}(m-1)m!\omega_n'(x_k)^{m-2}\omega_n''(x_k)$$

and

$$[\omega_n^{m-1}(x)]_{x=x_k}^{(m-1)} = (m-1)!\omega_n'(x_k)^{m-1}.$$

Then (3.10) become

$$\frac{1}{2}(m-1)\omega_n''(x_k)q_i(x_k) + \omega_n'(x_k)q_i'(x_k) = \frac{\delta_{ik}}{m!\omega_n'(x_k)^{m-2}}, \quad k = 1, 2, \cdots, n. \tag{3.11}$$

It follows from (3.2) and (3.4) that

$$\omega_n'(x_k) = \begin{cases} \gamma_0, & k = 1 \\ x_k L_{n-1}^{\prime(\alpha)}(x_k), & k > 1 \end{cases} \tag{3.12}$$

and

$$\omega_n''(x_k) = \begin{cases} 2\gamma_1, & k = 1 \\ (x_k - \alpha + 1) L_{n-1}^{\prime(\alpha)}(x_k), & k > 1. \end{cases} \tag{3.13}$$

These, together with (3.11), give

$$(\alpha + 1)q_i'(0) - (n-1)(m-1)q_i(0) = \frac{(\alpha+1)\delta_{i1}}{m!\gamma_0^{m-1}} \tag{3.14}$$

and

$$x_k q_i'(x_k) + (\frac{m-1}{2}x_k - \gamma)q_i(x_k) = \frac{x_k\delta_{ik}}{m!\omega_n'(x_k)^{m-1}}, \quad k = 2, \cdots, n$$

or

$$x_k q_i'(x_k) + (\frac{m-1}{2}x_k - \gamma)q_i(x_k) = \frac{x_i\delta_{ik}}{m!\omega_n'(x_i)^{m-1}}, \quad k = 2, \cdots, n. \tag{3.15}$$

Denote by $\mathbf{D}$ the differential operator

$$\mathbf{D}y := xy' + (\frac{m-1}{2}x - \gamma)y.$$

Then (3.15) implies

$$\mathbf{D}q_i(x) = Q_i(x) - (c_i + e_i x)L_{n-1}^{(\alpha)}(x), \tag{3.16}$$

where c_i and e_i are constants to be determined and $Q_i(x)$ is given by (3.8). Solving this differential equation we get (3.7) with a constant d_i to be determined.

Now let us determine c_i, e_i and d_i. To this end put

$$q_i(x) = \sum_{k=0}^{n-1} \alpha_k x^k. \tag{3.17}$$

Meanwhile we write

$$Q_i(x) = \sum_{k=0}^{n-1} \beta_k x^k. \tag{3.18}$$

Comparing the coefficients of x^k on both sides in (3.16) by means of (3.17),(3.18), and (3.3) and adding (3.14), we obtain the system of equations with the unknowns $\alpha_0, \cdots, \alpha_{n-1}, c_i, e_i$

$$\begin{cases} (n-1)(m-1)\alpha_0 - (\alpha+1)\alpha_1 = -\frac{(\alpha+1)\delta_{i1}}{m!\gamma_0^{m-1}} \\ \frac{m-1}{2}\alpha_{k-1} + (k-\gamma)\alpha_k + \gamma_k c_i + \gamma_{k-1} e_i = \beta_k, \quad k = 0, 1, \cdots, n \\ \alpha_{-1} = \alpha_n = \gamma_{-1} = \gamma_n = \beta_n = 0. \end{cases} \tag{3.19}$$

The coefficient determinant of this system is

$$D_n(\alpha) = \begin{vmatrix} (n-1)(m-1) & -(\alpha+1) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\gamma & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_0 & 0 \\ \frac{m-1}{2} & 1-\gamma & 0 & 0 & \cdots & 0 & 0 & \gamma_1 & \gamma_0 \\ 0 & \frac{m-1}{2} & 2-\gamma & 0 & \cdots & 0 & 0 & \gamma_2 & \gamma_1 \\ 0 & 0 & \frac{m-1}{2} & 3-\gamma & \cdots & 0 & 0 & \gamma_3 & \gamma_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{m-1}{2} & n-1-\gamma & \gamma_{n-1} & \gamma_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{m-1}{2} & 0 & \gamma_{n-1} \end{vmatrix}$$

Expanding this determinant by Lapace's Theorem in terms of the elements of the first two columns and the last two columns we get

$$\begin{aligned} D_n(\alpha) &= \left(\frac{m-1}{2}\right)^{n-2} \begin{vmatrix} (n-1)(m-1) & -(\alpha+1) & 0 & 0 \\ -\gamma & 0 & \gamma_0 & 0 \\ \frac{m-1}{2} & 1-\gamma & \gamma_1 & \gamma_0 \\ 0 & \frac{m-1}{2} & \gamma_2 & \gamma_1 \end{vmatrix} \\ &+ \sum_{k=2}^{n-1} \left(\frac{m-1}{2}\right)^{n-1-k} (k-1)! \begin{pmatrix} \gamma-2 \\ k-1 \end{pmatrix} \begin{vmatrix} (n-1)(m-1) & -(\alpha+1) & 0 & 0 \\ -\gamma & 0 & \gamma_0 & 0 \\ \frac{m-1}{2} & 1-\gamma & \gamma_1 & \gamma_0 \\ 0 & 0 & \gamma_{k+1} & \gamma_k \end{vmatrix} \\ &= 2\gamma_0^2(n-1)\left(\frac{m-1}{2}\right)^n + \gamma_0 \sum_{k=1}^{n-1} \left(\frac{m-1}{2}\right)^{n-1-k} (k-1)! \begin{pmatrix} \gamma-2 \\ k-1 \end{pmatrix} \gamma_k \\ &\quad \times \left\{\gamma(\alpha+1)\frac{\gamma_{k+1}}{\gamma_k} - \gamma(\alpha+1)\frac{\gamma_1}{\gamma_0} - \frac{1}{2}(m-1)(\alpha+1) - (1-\gamma)(n-1)(m-1)\right\}. \end{aligned}$$

Using the relations

$$\gamma_{k+1} = -\frac{n-1}{(k+1)(k+1+\alpha)}\gamma_k, \quad k = 0, 1, \cdots, n-1$$

the expression in the curved brackets is equal to

$$-\frac{\gamma(\alpha+1)(n-1-k)}{(k+1)(k+1+\alpha)}+\gamma(n-1)-\frac{1}{2}(m-1)(\alpha+1)-(1-\gamma)(n-1)(m-1)$$
$$=\tau-\frac{\gamma(\alpha+1)(n-1-k)}{(k+1)(k+1+\alpha)}.$$

We know that the system (3.19) has a unique solution if and only if (3.5) is true. By Lemma this is equivalent to the regularity of $(0,1,\cdots,m-2,m)$ interpolation.

Solving (3.19) by Cramer's rule we get c_i and e_i. As for d_i we note that by (3.7) and (3.17)

$$d_i=e^{\frac{m-1}{2}}q_i(1)=e^{\frac{m-1}{2}}\sum_{k=0}^{n-1}\alpha_k.$$

To calculate α_k we use (3.19) to get

$$\alpha_{k-1}=\frac{2}{m-1}\{\beta_k-\gamma_k c_i-\gamma_{k-1}e_i+(\gamma-k)\alpha_k\},\quad k=n,n-1,\cdots,1. \tag{3.20}$$

This completes the proof.

Although Theorem 1 gives a necessary and sufficient condition of regularity in a manageable form, it does not provide a practical information of regularity on n and α. The next theorem provides a sufficient condition of this type in which $[\gamma]$ stands for the largest integer not more than γ.

Theorem 2 *The problem of $(0,1,\cdots,m-2,m)$ interpolation on the zeros of $xL_{n-1}^{(\alpha)}(x)(\alpha>-1)$ is regular if one of the following conditions is satisfied:*

(a) $\gamma\geq 2$ *and*

$$n>\begin{cases}\frac{1}{2}(\gamma-1)\{(m+1)\gamma+2\}, & \gamma=[\gamma]\\ \frac{[\gamma]\{(m-3)[\gamma]+4\gamma+m-1\}}{2(\gamma-[\gamma])}, & \gamma\neq[\gamma];\end{cases} \tag{3.21}$$

(b) $2>\gamma>\frac{2(m-1)}{2m-1}$ *and*

$$n\geq\frac{2(2m-1)\gamma+2(m-1)^2}{(2m-1)\gamma-2(m-1)}; \tag{3.22}$$

(c) $\gamma\leq\frac{m-1}{m}$

Proof For simplicity we introduce the following symbols

$$\delta_k=\tau-\frac{\gamma(\alpha+1)(n-1-k)}{(k+1)(k+1+\alpha)},\quad k=1,\cdots,n-1, \tag{3.23}$$

$$\begin{cases}a_0=2\gamma_0^2(n-1)(\frac{m-1}{2})^n,\\ a_k=\frac{\gamma_0}{k}(\frac{m-1}{2})^{n-1-k}\binom{n-1+\alpha}{n-1-k}\binom{\gamma-2}{k-1}\delta_k,\quad k=1,\cdots,n-1.\end{cases} \tag{3.24}$$

Then

$$a_k = \frac{2(n-k)(\gamma-k)\delta_k}{(m-1)k(k+\alpha)\delta_{k-1}}a_{k-1} := \tau_k\frac{\delta_k}{\delta_{k-1}}a_{k-1}, \quad k=2,\cdots,n-1. \tag{3.25}$$

(a) As preliminaries, first we show some claims.

Claim 1. $\dfrac{[\gamma]\{(m-3)[\gamma]+4\gamma+m-1\}}{2(\gamma-[\gamma])} > \dfrac{1}{2}(\gamma-1)\{(m+1)\gamma+2\}$.

In fact ,we have

$$\begin{aligned}&[\gamma]\{(m-3)[\gamma]+4\gamma+m-1\}-(\gamma-1)\{(m+1)\gamma+2\}\\&=([\gamma]-\gamma)\{(m-3)([\gamma]+\gamma)+4\gamma\}+(m-1)([\gamma]+\gamma)+2\\&>-\{(m-3)([\gamma]+\gamma)+4\gamma\}+(m-1)([\gamma]+\gamma)+2\\&=2([\gamma]+1-\gamma)>0,\end{aligned}$$

which proves our chain.

Claim 2. $a_1 > a_0$.

In fact, under the assumptions and using (3.21) one has

$$\begin{aligned}\frac{a_1}{a_0} &= \frac{2\tau-\frac{\gamma(\alpha+1)(n-2)}{\alpha+2}}{(m-1)^2(\alpha+1)} > \frac{2\tau-\gamma(n-2)}{(m-1)^2(\alpha+1)} = \frac{\{(2m-1)\gamma-2(m-1)\}n-2m\gamma}{2(m-1)(\gamma+m-1)}\\ &> \frac{\{(2m-1)\gamma-2(m-1)\}(\gamma-1)\{(m+1)\gamma+2\}-4m\gamma}{4(m-1)(\gamma+m-1)}\\ &\geq \frac{m\{(m+1)\gamma+2\}-2m\gamma}{2(m-1)(\gamma+m-1)} > 1,\end{aligned}$$

which proves our claim.

In this case we see that $n \geq [\gamma]$,

$$\delta_k > \delta_{k-1} > 0, \quad k=2,\cdots,n-1 \tag{3.26}$$

and $a_k > 0, 0 \leq k < \gamma$.

Then we note

$$\tau_k < \tau_{k-1}, \quad 3 \leq k < \gamma. \tag{3.27}$$

Meanwhile it is easy to check that (3.21) is equivalent to that $\tau_{\gamma-1} > 1$ for $\gamma = [\gamma]$ and $\tau_{[\tau]} > 1$ for $\gamma \neq [\gamma]$. Thus we have

$$a_k > a_{k-1} > 0, \quad 1 \leq k < \gamma. \tag{3.28}$$

For the proof proper of the theorem if γ is an integer then

$$|D_n(\alpha)| = \left|\sum_{k=0}^{\gamma-1}(-1)^k a_k\right| \geq a_{\gamma-1} - a_{\gamma-2} > 0.$$

If γ is not an integer then

$$\operatorname{sgn} a_k = (-1)^{k+[\gamma]}, \quad k=[\gamma],[\gamma]+1,\cdots,n-1$$

and hence

$$|D_n(\alpha)| = \left|\sum_{k=0}^{n-1}(-1)^k a_k\right| \geq \sum_{k=[\gamma]}^{n-1}|a_k| - \left|\sum_{k=0}^{[\gamma]-1}(-1)^k a_k\right| \geq a_{[\gamma]} - a_{[\gamma]-1} > 0.$$

(b) In this case by a similar proof as before we see

$$\frac{a_1}{a_0} > \frac{\{(2m-1)\gamma - 2(m-1)\}n - 2m\gamma}{2(m-1)(\gamma+m-1)} \geq 1.$$

Hence (3.26) still holds and $(-1)^k a_k \leq 0, k = 1, 2, \cdots, n-1$. Thus $D_n(\alpha) < 0$.

(c) In this case we see that for $k = 1, 2, \cdots, n-1$

$$\begin{aligned} \delta_k &\leq \gamma = (mn - m - 1)\gamma - n(m-1) \leq \frac{(m-1)(mn-m-1)}{m} - n(m-1) \\ &= \frac{1-m^2}{m} < 0 \end{aligned}$$

and hence $(-1)^k a_k > 0, k = 0, 1, \cdots, n-1$, which implies $D_n(\alpha) > 0$.

This completes the proof.

Theorem 3 *If the problem of $(0, 1, \cdots, m-2, m)$ interpolation on the zeros of $\omega_n(x) := xL_{n-1}^{(\alpha)}(x)(\alpha > -1)$ is not regular, i.e., $D_n(\alpha) = 0$, then the general form of the solution of (1.3) with $y_{kj} \equiv 0$ is*

$$R_{mn-1}(x) = C\omega_n^{m-1}(x)q(x) \tag{3.29}$$

in which C is an arbitrary number and $q \in \mathbf{P}_{n-1}$ is of the form

$$q(x) = x^\gamma e^{-\frac{(m-1)x}{2}}\{d + \int_1^x [\gamma(\alpha+1) + \tau t]L_{n-1}^{(\alpha)}(t)t^{-\gamma-1}e^{\frac{(m-1)t}{2}}dt\} \tag{3.30}$$

with a certain constant d.

Proof Obviously, $R_{mn-1} \in \mathbf{P}_{mn-1}$ is a solution of (1.3) with $y_{kj} \equiv 0$ if and only if $R_{mn-1}(x)$ is of the form

$$R_{mn-1}(x) = \omega_n^{m-1}(x)q(x), \quad q \in \mathbf{P}_{n-1} \tag{3.31}$$

and satisfies

$$[\omega_n^{m-1}(x)q(x)]|_{x=x_k}^{(m)} = 0, \quad k = 1, 2, \cdots, n. \tag{3.32}$$

Comparing (3.32) with (3.10) and following the line of the proof of Theorem 1 we can show that $q = \sum_{k=0}^{n-1}\alpha_k x^k$ satisfies the differential equation with arbitrary numbers C and E

$$\mathbf{D}q(x) = (C + Ex)L_{n-1}^{(\alpha)}(x) \tag{3.33}$$

and the system of equations

$$\begin{cases} (n-1)(m-1)\alpha_0 - (\alpha+1)\alpha_1 = 0 \\ \frac{m-1}{2}\alpha_{k-1} + (k-\gamma)\alpha_k = C\gamma_k + E\gamma_{k-1}, \quad k = 0, 1, \cdots, n \\ \alpha_{-1} = \alpha_n = \gamma_{-1} = \gamma_n = 0, \end{cases} \tag{3.34}$$

which are analogues of (3.16) and (3.19), respectively. Moreover, if we can show that the equation (3.33) with $C=\gamma(\alpha+1)$ has a unique solution (3.30) then the proof of the theorem is complete. Solving (3.33) with $C=\gamma(\alpha+1)$ we get

$$q(x)=x^{\gamma}e^{-\frac{(m-1)x}{2}}\{d+\int_1^x[\gamma(\alpha+1)+Et]L_{n-1}^{(\alpha)}(t)t^{-\gamma-1}e^{\frac{(m-1)t}{2}}dt\}$$

with constants d and E to be determined.

To determine d we note

$$d=e^{\frac{m-1}{2}}\sum_{k=0}^{n-1}\alpha_k.$$

Obviously, $D_n(\alpha)=0$ means that (3.34) has a nontrivial solution $\alpha_0,\cdots,\alpha_{n-1},C,E$. In this case we see that if $\gamma=0$ then $C=0$ follows from the equation corresponding to $k=0$ in (3.34) and if $\gamma\neq 0$ then $C\neq 0$, for otherwise solving the first three equations in (3.34) yields

$$\alpha_0=-\frac{C\gamma_0}{\gamma},\quad \alpha_1=-\frac{C\gamma_0(n-1)(m-1)}{\gamma(\alpha+1)},$$

and

$$E=\frac{1}{\gamma_0}\{\frac{m-1}{2}\alpha_0+(1-\gamma)\alpha_1-C\gamma_1\}=\frac{C\tau}{\gamma(\alpha+1)}. \tag{3.35}$$

Hence $\alpha_0=\cdots=\alpha_{n-1}=C=E=0$ would occur, which is impossible. This shows that the system (3.34) with $C=\gamma(\alpha+1)$ must have a solution. On the other hand, $C=\gamma(\alpha+1)$ implies by (3.35) that $E=\tau$, and from (3.34) we can uniquely solve

$$\alpha_{k-1}=\frac{2}{m-1}\{\gamma(\alpha+1)\gamma_k+\tau\gamma_{k-1}+(\gamma-k)\alpha_k\},\quad k=n,n-1,\cdots,1$$

and hence uniquely determine d.

This complete the proof.

References

[1] J.Prasad and R.B.Saxena, *Some interpolatory properties of Laguerre polynomials*, Canad. Math. Bull., **10**(1967), 559–571.

[2] A.M.Chak and J.Szabados, *On (0,2) interpolation for the Laguerre abscissas*, Acta Math. Hungar., **49(3–4)**(1987), 415–423.

[3] Shi Yingguang, *Regularity and explicit representation of* $(0,1,\cdots,m-2,m)$*-interpolation on the zeros of the Jacobi polynomials* , J. Approx. Theory, **76**(1994), 274–285.