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Oscillation of Solutions for a Second Order Nonlinear Perturbed Differential Equations with Functional Arguments *

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Abstract Conditions are given on the functions f, g, h, p and q which imply that all continuable solutions of

$$\ddot{x} + p(t)f(\dot{x}) + q(t)g(x) = h(t, x, \dot{x})$$

are abounded as well as oscillatory on the interval $[t_0, \infty), t_0 > 0$.

Keywords nonlinear differential equation, oscillation, continuable solution.

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1. Introduction

In this paper we consider second order nonlinear ordinary differential equation with functional coefficients

$$\ddot{x}(t) + p(t)f(\dot{x}(t)) + q(t)g(x(t)) = h(t, x(t), \dot{x}(t)), \quad (. = d/dt). \tag{1}$$

This equation describes a peturbed physical system with nonlinear time dependent restoring force, as well as nonlinear time dependent damping. Positive damping is commonly encountered in application, but negative damping or damping of variable sign is also found: for example, the well known Van der Pol Oscillator

$$\ddot{x} + \lambda(x^2 - 1)\dot{x} + wx = 0,$$

and this equation with variable coefficients λ and w (corresponding, e.g. to an inductively tuned troide oscillator [9]. We consider the problem of giving conditions on the functions f, g, h, p and q for which all continuable solutions of (1) are bounded as well as oscillatory. By continuable, we mean a solution which is defined on the interval $[t_0, \infty)$. A solution is said to be oscillatory if, given $t_1 > 0$, there is $t > t_1$ such that $x(t) = 0$; the equation itself is said to be oscillatory if all its solutions are oscillatory.

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In the absence of the damping and the external force, there is a very large body of literature devoted to the corresponding equation

$$\ddot{x} + q(t)g(t)) = 0. \tag{2}$$

we refer to [12] for an excellent and comprehensive bibiliography until 1968.

The use of averaging functions in the study of oscillations dates back to Wintner[11] and Hartman[6]. Coles[3] and Willett[9], and more recently, Kwong and Zettl[7] developed averaging techniques and respectively established more general theorems for equation (2) and for the more general equation

$$(v(t)\dot{x}(t)) + q(t)x(t) = 0. \tag{3}$$

$v(t) > 0$, by considering weighted averages of the integral of $q(t)$.

The problem of finding criteria for the oscillation of equation (3) has been studied recently by various authors including Grace[4], Philos[8] and Yan[13]. For equations in which the perturbation term $h(t,x,y)$ depends on x, such as equation (1), realatively few oscillation criteria are known. The purpose of this paper is to develop some oscillation criteria for the differential equation (1).

In the first section, we give conditions under which all solutions of equation (1) are bounded for $t \in [t_0, \infty)$, and in the second section we give criterion in order that all bounded solutions of (1) oscillate. That our results are fairly sharp will be illustrated by some examples.

In the sequel we assume that

(i) $f : R \to R$ is continuous and $uf(u) > 0$ for all $u \neq 0$.

(ii) $g : R \to R$ is differentiable such that $xg(x) > 0$ and $g'(x) = \frac{d}{dx}g(x) > 0$ for all $x \neq 0$.

(iii) $h : [t_0, \infty) \times R \times R \to R$ is continuous such that $h(t,x,y) \leq \phi(t)\varepsilon(x)\rho(y)$, where $\varepsilon, \rho : R \to R$ are bounded and continuous such that $\varepsilon(x) \leq g(x)$ for all $x \neq 0$, and $\phi : [t_0, \infty) \to [0, \infty)$ is continuous.

(iv) $p : [t_0, \infty) \to R$ is continuous.

(v) $q : [t_0, \infty) \to (0, \infty)$ is differentiable.

2. Boundedness of Solutions

Theorem 1 *If in addition to conditions (i)–(v) above, we assume that*

(vi) $P(t) \geq 0$ *for all* $t \geq t_0 > 0$. *There exists a positive function* $\psi(t)$ *such that*

(vii) $\int_{t_0}^{\infty} \frac{\phi(t)}{q(t)\psi(t)} dt < 0$, *and*

(viii) $\int_{t_0}^{\infty} |c_1\phi(t)\psi(t) - \frac{q'(t)}{q(t)}| dt < \infty$, *for some constant* c_1.

Then all continuable solutions of (1) are bounded.

Proof Let $x(t)$ be a regular solution of equation (1) for $t \geq t_0 > 0$. Multiplying equation (1) by $\dot{x}(t)/q(t)$, yields

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\dot{x}^2(t)}{q(t)}\right) + \frac{1}{2}\frac{q'(t)\dot{x}^2(t)}{q^2(t)} + \frac{p(t)f(\dot{x}(t))\dot{x}(t)}{q(t)} + g(x(t)) \leq c_1\phi(t)\frac{|\dot{x}(t)|}{q(t)} \tag{4}$$

where c_1 is constant.

Integrating (4) from t_0 to t we obtain

$$\frac{\dot{x}^2(t)}{2q(t)} + G(x(t)) \le c_2 - \frac{1}{2}\int_{t_0}^{t} \frac{q'(x)}{q^2(s)} x^2(x)ds - \int_{t_0}^{t} \frac{p(s)}{q(s)} f(\dot{x}(s))\dot{x}(s)ds + c_1 \int_{t_0}^{t} \frac{\phi(s)}{q(s)} |\dot{x}(s)|ds, \tag{5}$$

where $G(x) = \int_0^x g(s)ds$ and $c_2 = \dot{x}^2(t_0)/(2q(t_0)) + G(x(t_0))$.

By virtue of condition (i), (v) and (vi), the above inequatlity becomes

$$\frac{\dot{x}^2(t)}{2q(t)} + G(x(t)) \le c_2 - \frac{1}{2}\int_{t_0}^{t} \frac{q'(x)}{q^2(s)} \dot{x}^2(x)ds + c_1 \int_{t_0}^{t} \frac{\phi(s)}{q(s)} |\dot{x}(s)|ds, \tag{6}$$

The inequality $2ab \le a^2/\sigma(t) + \sigma(t)b^2$; where σ is any positive function, implies that

$$\int_{t_0}^{t} \frac{\phi(s)}{q(s)} |\dot{x}(s)|ds \le \int_{t_0}^{t} \frac{\phi(s)}{2q(s)} [\psi(s)\dot{x}^2(s) + \frac{1}{\psi(s)}]ds.$$

Therefore (6) become

$$\frac{\dot{x}^2(t)}{2q(t)} + G(x(t)) \le c_2 + \int_{t_0}^{t} \frac{\dot{x}^2(s)}{2q(s)} [c_1\phi(s)\psi(s) - \frac{q'(s)}{q(s)}]ds + \frac{c_1}{2}\int_{t_0}^{t} \frac{\phi(s)}{q(s)\psi(s)} ds. \tag{7}$$

By virtue of (v),(vi) and (vii) the above inequality (7) becomes

$$\begin{aligned} \frac{\dot{x}^2(t)}{2q(t)} + G(x(t)) &\le M + \int_{t_0}^{t} \frac{\dot{x}^2(s)}{2q(s)} |c_1\phi(s)\psi(s) - \frac{q'(s)}{q(s)}|ds \\ &\le M + \int_{t_0}^{t} \left(\frac{\dot{x}^2(s)}{2q(s)} + G(x(s))\right) |c_1\phi(s)\psi(s) - \frac{q'(s)}{q(s)}|ds \end{aligned} \tag{8}$$

as $G(x) \ge 0$ by the condition (ii), where

$$M = c_2 + \frac{c_1}{2}\int_{t_0}^{\infty} \frac{\phi(s)}{q(s)\psi(s)} ds.$$

By the Gronwall inequality, we have that

$$\frac{\dot{x}^2(t)}{2q(t)} + G(x(t)) \le M \exp \int_{t_0}^{\infty} |c_1\phi(c)\psi(s) - \frac{q'(s)}{q(s)}|ds < \infty$$

Since $g'(x) > 0$ for $x \ne 0, x(t)$ should be uniformly bounded. Moreover, if q is bounded, then $\dot{x}(t)$ is also bounded. This completes the proof. □

Remark 1 In the previous discussion it has been assumed that the function p was positive. Indeed, our method can be applied to certain class of equations (1) with $p(t)$ negative. We rewrite equation (1) in the following form

$$\ddot{x} + e(t)f(\dot{x}) + q(t)g(x) = (e(t) - p(t)f(\dot{x}) + h(t, x, \dot{x}),$$

where $e(t)$ is a positive function.

Theorem 2 *Assume, in addition to the conditions (i)–(v), that*

(ix) $p(t) \leq 0$ *for all* $t \geq t_0$*, there exists a positive continuous function* $e(t)$ *such that*

(x) $\int_{t_0}^{\infty} \frac{e(t) - p(t)}{q(t)} dt < \infty$,

(xi) $\int_{t_0}^{\infty} |\{c_1\phi(t)\psi(t) - \frac{q'(t)}{q(t)} + (e(t) - p(t))\}| dt < \infty$.

(xii) $f^2(u) \leq \alpha u^2 + \beta; \alpha > 0$ *and* $\beta \geq 0$.

Then all continuable solutions of (1) are bounded.

Proof Using reasoning analogous to that used in the proof of Theorem 1, we must estimate the expression

$$\int_{t_0}^{t} \frac{e(s) - p(s)}{q(s)} f(\dot{x}(s))\dot{x}(s) ds$$

This integral satisfies

$$\int_{t_0}^{t} \frac{e(s) - p(s)}{q(s)} f(\dot{x}(s))\dot{x}(s) ds \leq \int_{t_0}^{t} \frac{e(s) - p(s)}{2q(s)} [\dot{x}^2(s) + f^2(\dot{x}(s))] ds.$$

Now, since $f^2(u) \leq \alpha u^2 + \beta; \alpha > 0$ and $\beta \geq 0$, then we can improve $\frac{\dot{x}^2(t)}{2q(t)} + G(x(t))$ in (8). Then following the same procedure of Theorem 1 we have the same conclusion.

Remark 2 In fact, our method can be applied to a certain class of differential equations (1) in which $p(t)$ is allowed to change sign. This can be done by rewriting equation (1) as

$$\ddot{x} + p_+(t)f(\dot{x}) + q(t)g(x) = -p_-(t)f(\dot{x}) + h(t, x, \dot{x}) \qquad (1)'$$

where $p_+(t) = \max\{p(t), 0\}, p_-(t) = \min\{p(t), 0\}$. Then we have

Theorem 3 *Assume, in addition to the condition (i)–(v), that*

(xiii) $\int_{t_0}^{\infty} \frac{-p_-(t)}{q(t)} dt < \infty$,

(xiv) $\int_{t_0}^{\infty} |\{c_1\phi(t)\psi(t) - \frac{q'(t)}{q(t)} - p_-(t)\}| dt < \infty$,

(xv) $f^2(u) \leq \alpha u^2 + \beta, \alpha > 0, \beta \geq 0$.

Then all continuable solutions of (1) are bounded.

Proof The proof of Theorem 3 can be modelled on that of Theorem 2 and hence the proof is omitted.

3. Oscillations of the Solutions

In this section we show that all bounded solutions studied in section 2 are oscillatory.

The following result is concerned with the oscillation of the bounded solutions of equation (1) when $p(t) \geq 0$.

Theorem 4 *If in addition to the conditions (i)–(v), we assume that* *(xvi)* $p(t)\geq 0$ *and* $\int_{t_0}^{\infty} p(t)dt < \infty$,

(xvii) $\int_{t_0}^{\infty}(q(s)-\phi(s))ds=\infty$,

(xviii) $\lim_{t\to\infty}\int_{t_0}^{t}\int_{t_0}^{s}(q(u)-\phi(u))duds=\infty$.

Then all bounded solutions of (1) are oscillatory.

Proof Assume the contrary that there exists a solution $x(t)$ of (1) which satisfies $-m \leq x(t) \leq M, -\alpha < \dot{x}(t) < \beta$, where m, M, α and β are positive constants that is not oscillatory. Let $x(t)\geq 0$ for $t\geq T_1\geq t_0>0$. Without loss of generality we assume that $x(t)>0$ for $t\geq T_1$ (the case $x(t)<0$ can be treated similarly). Then

$$(\frac{\dot{x}(t)}{g(x(t))})'=\frac{\ddot{x}(t)}{g(x(t))}-\frac{\dot{x}^2g'(x(t))}{g^2(x(t))}\leq\frac{\ddot{x}(t)}{g(x(t))}$$

Substitute in equation, we have that

$$(\frac{\dot{x}(t)}{g(x(t))})'\leq\phi(t)-p(t)\frac{f(\dot{x}(t))}{g(x(t))}-q(t). \tag{9}$$

For $T_2\geq T_1$ an integration of (9) gives

$$\frac{\dot{x}(t)}{g(x(t))}\leq\frac{\dot{x}(T_2)}{g(x(T_2))}+\int_{T_2}^{t}(\phi(s)-q(s))ds-\int_{T_2}^{t}p(s)\frac{f(\dot{x}(s))}{g(x(s))}ds. \tag{10}$$

Now, if $\dot{x}(t)\geq 0$ for all $t\geq T_2$, then we have that

$$\int_{T_2}^{t}(q(s)-\phi(s))ds\leq\frac{\dot{x}(T_2)}{g(x(T_2))}. \tag{11}$$

Since $g'(x)>0$ for all $x\neq 0, g(x(t))\geq g(x(T_2))$ for all $t\geq T_2$. Hence, for all $t\geq T_2$

$$\int_{t}^{\infty}[q(s)-\phi(s)]ds\leq\frac{\dot{x}(t)}{g(x(t))}$$

and integrating we obtain

$$\int_{T_2}^{t}\int_{s}^{\infty}[q(u)-\phi(u)]duds\leq\int_{T_2}^{\infty}\frac{\dot{x}(s)}{g(x(s))}ds=\int_{x(T_2)}^{x(t)}\frac{du}{g(u)}.$$

Taking the limit as $t\to\infty$ we must have $x(t)\to\infty$ as $t\to\infty$, a contradiction to the boundedness of $x(t)$.

If $\dot{x}(t)<0$ for all $t\geq T_2\geq t_0$, then we have from (10) that

$$\int_{T_2}^{t}(q(s)-\phi(s))ds\leq k+K\int_{T_2}^{t}p(s)ds \tag{12}$$

since $x(t)$ and $\dot{x}(t)$ are bounded. Where k and K are constants. Taking the limit as $t\to\infty$, we see that the integral on the right side of (12) does not exist. This obvious contradiction completes the proof.

Example Consider the differential equation

$$\ddot{x}+\frac{1}{t^2}(\dot{x}+\dot{x}^3)+e^{-1/t}x=0,\quad t\geq 1. \tag{13}$$

One can see that all conditions of Theorems 1 and 4 are satisfied. Therefore all continuable solution of (13) are bounded oscillatory.

Note that none of the known criteria [1],[2] and [3] can apply to this equation.

The following result is concerned with the oscillation of the bounded solutions of (1) when $p(t)\leq 0$.

Theorem 5 *If in additions to the conditions (i)–(v), we assume that*

(xix) $h(t,x,y)=0$ *for all* t,x *and* $y=0$,

(xx) $p(t)\leq 0$ *for all* $t\geq t_0$,

(xxi) $\int_{t_0}^{\infty}(q(s)-\phi(s))ds=\infty$.

Then all continuable solutions of (1) such that $-A\leq x(t)\leq B$ *are oscillatory, where* A *and* B *are positive constants.*

Proof Suppose the theorem is false, then is there exists a solution of (1) satisfies $-A\leq x(t)\leq B$ and which is not oscillatory. Without loss of generality, we can assume that $x(t)>0$ for $t\geq T_1\geq t_0>0$. (A similar proof will apply if $x(t)<0$ for $t\geq T_1\geq t_0$). We proceed as in Theorem 4, and we have for $t\geq T_2$ the inequality (10). Now, let $t_\nu\in[T_1,\infty)$ be a cirtical point of $x(t)$, then equation (1) implies that $\ddot{x}(t_\nu)<0$. Therefore, t_ν is a maximum of $x(t)$. It follows that $x(t)$ is eventually monotone on $[T_2,\infty)\subset[T_1,\infty)$. Therefore, $\dot{x}(t)<0$ for all $t\in[T_2,\infty)$.

Hence, by virtue of the condition (i),(ii) and (xx) we have from (10) that

$$\frac{\dot{x}(t)}{g(x(t))}\leq\frac{\dot{x}(T_2)}{g(x(T_2))}+\int_{T_2}^{t}(\phi(s)-q(s))ds. \tag{14}$$

Since $x(t)$ is bounded, we have that $\dot{x}(t)$ is negative increasing and $\lim_{t\to\infty}\dot{x}(t)=0$. Hence, it follows from (14) that

$$\int_{T_2}^{\infty}(q(s)-\phi(s))ds\leq\dot{x}(T_2)/g(x(T_2)).$$

This obvious contradiction completes the proof.

Example Consider the differential equation

$$\ddot{x}-\frac{\dot{x}}{t}+4t^2x=0,\quad t\geq 1. \tag{15}$$

This equation is a special case of the so-called Emden-Fowler equation.

One can see easily that all conditions of Theorem 2 are satisfied, where $e(t)$ can be taken as $e(t)=\frac{1}{t}$. Also, all conditions of Theorem 5 are satisifed. Therefore, all continuable solutions of (15) are bounded as well as oscillatory. One such solution is $x(t) = \sin(t^2), t \geq 1$. This result can not be obtained from [2].

We conclude this paper by the following result, which is concerned with the oscillation of the bounded solutions of (1) when $p(t)$ may chang sign.

Consider equation (1) in its equivalent form (1)$'$. Let $H(t,x,y) = h(t,x,y) - p_-(t)f(\dot{x})$. Then we have

Theorem 6 *Assume, in addition to the condition (i)–(v), that*

(xxii) $\int_{t_0}^{\infty} p_+(t)dt < \infty$,

(xxiii) $H(t,x,y)/g(x) \leq \alpha F(t)$ *for* $t \in [t_0, \infty)$, *and* $|x| \leq A, |y| < B$. *Where* α *is constant and* $F : [t_0, \infty) \to (0, \infty)$ *is continuous,*

(xxiv) $\int_{t_0}^{\infty} (q(s) - \alpha F(s))ds = \infty$,

(xxv) $\int_{t_0}^{\infty} \int_{t_0}^{s} (q(u) - \alpha F(u))duds = \infty$.

Then all continuable solutions of (1) such that $|x(t)| \leq A, |\dot{x}(t)| < B$ *are oscillatory.*

Proof The proof of Theorem 6 can be modelled on that of Theorem 4 and hence is omitted.

Example Consider the differential equation

$$\ddot{x} + \frac{\cos t}{t^2}\dot{x} + x = 0, \quad t \geq 1.$$

Rewriting this equation as

$$\ddot{x} + \frac{\cos t}{t^2}\dot{x} + x = -\frac{\cos t}{t^2}\dot{x},$$

where $(\cos t) = \max\{\cos t, 0\}, (\cos t)_- = \min\{\cos t, 0\}$. One can verify that all conditions of theorem 6 are satisfied. Therefore, all bounded solutions of (1) are oscillatory.

References

[1] N.P.Bhatia, *Some oscillation theorems for second order differential equations*, J. Math. Anal. Appl., **15**(1966), 442–446.

[2] L.E.Bobisud, *Oscillation of solutions of damped nonlinear equations*, SIAM J. Appl. Math. Vol.19, No.3(1970).

[3] W.J.Coles, *An oscillation criterion for second order ddifferential equation* , Proc. Amer. Math. Soc., **19**(1968), 755–759.

[4] S.R.Grace, *Oscillation criteria for second order differential equations with damping*, J. Austral. Math. Soc.(Series A) **49**(1990), 43–54.

[5] S.R.Grace and B.S.Lalli, *Oscillation theorems for certain second order perturbed nonlinear differential equations*, J. Math. Anal. Appl., **77**(1980), 205–214.

[6] P.Hartma, *On nonoscillatory linear differential equations of second order*, Amer. J. Math., **74**(1952), 389–400.

[7] M.K.Kwong and A.Zettl, *Asymptotically constant function and second order oscillation*, J. Math. Anal. Appl., **93**(1983), 475– 494.

[8] G.Philos, *Integral averages and second order superlinear oscillation*, Math. Nachr., **120**(1985), 127–138.

[9] B.Van der Pol, *Oscillation hysteresis in a simple triode generator*, Phil. Mag., **43**(1922), 700–719.

[10] D.Willett, *On the oscillation behavior of the solutions of second order linear differential equations*, Ann. Polon. Math., **21**(1969), 175–194.

[11] A.Wintner, *A criterion of oscillatory stability* , Appl. Math., **7**(1949), 115–117.

[12] J.S.Wong, *On second order nonlinear oscillations*, Funkcial. Ekvac., **11**(1969), 207–234.

[13] J.Yan, *A note on oscillation criterion for an equation with damping*, Proc. Amer. Math. Soc., **90**(1984), 278–280.