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Proof Let $A \in C^{r\times r}$. Assume that there exist A_1, A_2 and A_3 such that

$$A_0 = \begin{pmatrix} A & A_1 \\ A_2 & A_3 \end{pmatrix} \begin{matrix} r \\ n-r \end{matrix}$$

is normal. That is, $A_0 A_0^H = A_0^H A_0$. Consequently,

$$AA^H - A^H A = A_2^H A_2 - A_1 A_1^H.$$

Hence

$$\mathrm{rank}(AA^H - A^H A) \leq \mathrm{rank}(A_2^H A_2) + \mathrm{rank}(A_1 A_1^H) \leq n-r+n-r = 2(n-r).$$

For the second part of the theorem, if $A=0$, the conclusion holds trivially. Otherwise, let $\sigma = 1/\|A\|_2$. Under the assumed condition by Theorem 2, σA has a unitary dilation of order n. Thus A may be extended to the multiple of $\|A\|_2$ of some unitary matrix of order n, which is a normal matrix. □

References

[1] P.R.Halmos, *Normal dilations and extensions of operators*, Summa Brasiliensis Math., **2**(1950), 125–134.

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矩阵酉扩张的刻划

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摘　要

本文用奇异值刻划了具有指定阶酉扩张的矩阵类，同时讨论了矩阵的正规扩张问题.

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Characterization of the Unitary Dilation of Matrices *

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Abstract We characterize those matrices that have unitary dilations of a given order. The normal dilation of a matrix is also discussed.

Keywords unitary dilation, singular value.

Classification AMS(1991) 15A18, 65F15/CCL O151.21

1. Introduction

Given $A \in C^{r\times s}$ and integer $n \geq r, s$. What property should A have for it to be a submatrix of some unitary matrix of order n? In the case $r = s$ and $n = 2r$, Halmos[1] answered this question. The result may be proved succinctly with the singular value decomposition (SVD). This paper completely solves the problem and partially answers the similar question for normal matrices.

Lemma 1[1] *Let A be a square matrix of order r. If $\|A\|_2 \leq 1$, then there exist A_1, A_2 and $A_3 \in C^{r\times r}$ such that*

$$\begin{pmatrix} A & A_1 \\ A_2 & A_3 \end{pmatrix}$$

is unitary.

Definition 1 *A matrix A is said to have a unitary dilation of order n if A is a submatrix of some unitary matrix of order n.*

We denote by $S(A)$ the number of the singular values of A equal to 1. Write U_r for the set of all unitary matrices of order r. A^H represents the conjugate transpose of A. For convenience we first consider the case when A is square. Since the interchange of rows or columns does not change the unitarity, we may let A be the left-upper block.

2. Main Results

Theorem 1 *Given $n \geq r$, let $A \in M_r(C)$.*

i) If $2r \leq n$, then A has a unitary dilation of order n if and only if $\|A\|_2 \leq 1$.

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ii) If $2r > n$, then A has a unitary dilation of order n if and only if $\|A\|_2 \le 1$ and $S(A) \ge 2r - n$.

Proof i) Since the spectral norm of a matrix is not less than that of any of its submatrix, if A can be extended to a unitary matrix then $\|A\|_2 \le 1$. Conversely, assume that $\|A\|_2 \le 1$. By Lemma 1, A has a unitary dilation of order $2r$. Naturally it can be extended to a unitary matrix of order n since $2r \le n$.

ii) Suppose A is a submatrix of some matrix $\in U_n$. Then $\|A\|_2 \le 1$ and there exist A_1, A_2 and A_3 such that

$$\tilde{A} = \begin{pmatrix} A & A_1 \\ A_2 & A_3 \end{pmatrix} \in U_n.$$

As a result $AA^H - I_r - A_1A_1^H$ and thus

$$\lambda(AA^H) = \{1 - \lambda_i(A_1A_1^H) : i = 1, \cdots, r\}. \tag{1}$$

By Sylvester Theorem the nonzero eigenvalues of $A_1A_1^H$ and those of $A_1^HA_1$ are the same (the multiplicity being counted). But the order of $A_1^HA_1$ is $n - r$, so the number of the nonzero eigenvalues of $A_1A_1^H$ is at most $n - r$. From relation (1) we know the number of the eigenvalues of AA^H equal to 1 is at least $r - (n - r) = 2r - n$,i.e., $S(A) \ge 2r - n$.

Conversely, let $\|A\|_2 \le 1$ and $S(A) \ge 2r - n$. Then the SVD of A can be written as

$$UAV = \begin{pmatrix} \Gamma & 0 \\ 0 & I_{2r-n} \end{pmatrix}, \quad \Gamma = \mathrm{diag}(\gamma_i), 0 \le \gamma_i \le 1.$$

Let $\Sigma = \{I - \Gamma^2\}^{1/2} \in C^{(n-r)\times(n-r)}$. Set $A_1 = U^H \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} \begin{matrix} n-r \\ 2r-n \end{matrix}$, $A_2 = (-\Sigma_{n-r}, 0_{2r-n})V^H$ and $A_3 = \Gamma$. Then

$$\begin{pmatrix} A & A_1 \\ A_2 & A_3 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & I_{n-r} \end{pmatrix}^H \begin{pmatrix} \Gamma & 0 & \Sigma \\ 0 & I_{2r-n} & 0 \\ -\Sigma & 0 & \Gamma \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}^H$$

is a product of unitary matrices, a unitary matrix. □

Lemma 2 *Let $r \le s$, $F = \begin{pmatrix} A \\ B \end{pmatrix} \begin{matrix} r \\ s-r \end{matrix}$ and $\|A\|_2 \le 1$. Then*

$$S(F) - S(A) \le s - r.$$

Proof

$$FF^H = \begin{pmatrix} AA^H & AB^H \\ BA^H & BB^H \end{pmatrix}.$$

Denote the eigenvalues of FF^H and AA^H by $\lambda_1 \ge \cdots \ge \lambda_s$ and $\mu_1 \ge \cdots \ge \mu_r$, respectively. Let $S(A) = t$. Then $\mu_1 = \cdots = \mu_t = 1 > \mu_{t+1} \ge \cdots \ge \mu_r$. Applying Cauchy-Poincaré Theorem [2, p59] yields

$$\mu_i \ge \lambda_{i+(s-r)}, i = 1, 2, \cdots, r.$$

Since there are $r-t$ μ_i's<1, from the above inequality we know that there are at least $r-t$ λ_i's<1. The singular values of F are $\sigma_i(F)=\lambda_i^{1/2}, i=1,\cdots,s$. Thus $S(F)\le s-(r-t)$ and consequently $S(F)-S(A)\le s-(r-t)-t=s-r$. □

Theorem 2 *Let $A\in C^{r\times s}$ and $r,s\le n$. Then a necessary and sufficient condition for A to have a unitary dilation of order n is $\|A\|_2\le 1$ and $S(A)\ge r+s-n$.*

Proof Let $p=\max\{r,s\}$.

Case 1 $2p\le n$. The condition $S(A)\ge r+s-n$ always holds and the conclusion is obvious from Theorem 1.

Case 2 $2p>n$. Without losing of generality we suppose $r\le s$. If A has a unitary dilation of order n, then $\|A\|_2\le 1$ and there exists some $B\in C^{(s-r)\times s}$ such that $A_0=\begin{pmatrix}A\\B\end{pmatrix}$ has a unitary dilation of order n. By Theorem 1 we have $S(A_0)\ge 2s-n$. From Lemma 2 we know $S(A_0)-S(A)\le s-r$. Combining the above two inequalities gives

$$S(A)\ge S(A_0)-(s-r)\ge 2s-n-(s-r)=r+s-n.$$

Conversely, assume that $\|A\|_2\le 1$ and $S(A)\ge r+s-n$. Consider the SVD

$$A=U(\Sigma,0)V,$$

where $U\in U_r, V\in U_s, \Sigma=\mathrm{diag}(\sigma_i)$ and

$$\sigma_1=\cdots=\sigma_{r+s-n}=1\ge\cdots\ge\sigma_r\ge 0.$$

If $r+s<n$, then we need not know $S(A)$ in the following proof. Set $B=(0,I_{s-r})V$. Then

$$A_0=\begin{pmatrix}A\\B\end{pmatrix}=\mathrm{diag}(U,I)\begin{pmatrix}\Sigma&0\\0&I_{s-r}\end{pmatrix}V$$

satisfies $\|A_0\|_2\le 1$ and $S(A_0)\ge r+s-n+(s-r)=2s-n$. By Theorem 1 A_0 and thus A has a unitary dilation of order n. □

Definition 2 *A matrix A is said to have a normal dilation of order n if there exist matrices A_1, A_2, and A_3 such that*

$$\begin{pmatrix}A&A_1\\A_2&A_3\end{pmatrix}$$

is a normal matrix of order n.

Not every matrix may be dilated to a normal one of some designated order. For example,

$$A=\begin{pmatrix}1&0&1\\0&1&1\\1&0&0\end{pmatrix}$$

does not have any normal dilation of order 4. This can be seen from the next theorem and by noticing that

$$\operatorname{rank}(AA^H - A^H A) = 3 > 2(4-3).$$

How do we characterize those matrices that have normal dilations of a given order? This problem is much more difficult. We now answer it partly.

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