where $p(1,1) \neq 0$. If there exists a $\Delta_k \neq 0 (2 \leq k \leq n)$, then

$$v_1 = e_1 + \frac{p(1,k)+\sqrt{\Delta_k}}{2p(1,1)} e_k + \sum_{j \neq 1,k} \frac{p(1,j)\sqrt{\Delta_k} + p(1,k)p(1,j) - 2p(1,1)p(k,j)}{2p(1,1)\sqrt{\Delta_k}} e_j,$$

$$v_2 = p(1,1)e_1 + \frac{p(1,k)-\sqrt{\Delta_k}}{2} e_k + \sum_{j \neq 1,k} \frac{p(1,j)\sqrt{\Delta_k} - p(1,k)p(1,j) + 2p(1,1)p(k,j)}{2\sqrt{\Delta_k}} e_j,$$

otherwise,

$$v_1 = e_1 + \sum_{j=2}^{n} \frac{p(1,j)}{2p(1,1)} e_j, \quad v_2 = p(1,1)e_1 + \sum_{j=2}^{n} \frac{p(1,j)}{2} e_j,$$

where $\Delta_k = p(1,k)^2 - 4p(1,1)p(k,k)$.

References

[1] Wang Boying, *The Basic of Multilinear Algebra*, Beijing Normal University Press, 1985.

[2] Zhu Zhongnan, *A necessary and sufficient condition satisfied by decomposable elements in the $V^{(2)}$ completely symmetric tensor space*, Journal of Nanjing University (Natural Sciences Edition), Vol.25(1989), 705–710.

[3] R.Merris, *Equality of decomposable symmetric tensors* , Canal. J. Math., **27**(1975), 1022–1027.

[4] J.A.Dias da Silva, *Condition for equality of decomposable symmetric tensors* , Linear Algebra and Appl., **24**(1979), 85–92.

二阶完全对称张量空间中可分元素的坐标间的齐次关系

朱 忠 南

(江苏教育学院数学系，南京210013)

摘 要

本文给出了二阶完全对称张量空间$V^{(2)}$中可分元素的坐标间的齐次关系，特别地，这种齐次关系可以由行列式表示出来．并由此得到了$V^{(2)}$中可分元素的一个等价条件．

Journal of Mathematical Research & Exposition
Vol.15, No.3, 345–349, Aug. 1995

Homogenous Relations among the Coordinates of a Decomposable Element in 2^{nd} Completely Symmetric Tensor Space *

Zhu Zhongnan
(Jiangsu Education Institute, Nanjing 210013)

Abstract We get some homogenous relations among the coordinates of a decomposable element in $V^{(2)}$, the 2^{nd} completely symmetric tensor space. In particular, the homogenous expressions can be expressed by determiants. Thus, we obtain an equivalent condition of the decomposable element in $V^{(2)}$.

Keywords decomposable element, spanning vectors, homogenous relation.

Classification AMS(1991) 15A69/CCL O151.23

Let V be an n-dimensional vector space over a field R of characterstic zero. For $n \geq 2$, let $V^{(2)} = V \cdot V$ denote the 2^{nd} completely symmetric tensor space. Denote

$$G_{2,n} = \{\alpha = (i,j) | i,j \in Z, 1 \leq i \leq j \leq n\}.$$

If $\{e_1, \cdots, e_n\}$ is a basis of V, then $\{e^{\cdot}_{\alpha} = e^{\cdot}_{ij} | \alpha = (i,j) \in G_{2,n}\}$ is a basis of $V^{(2)}$, where $e^{\cdot}_{ij} = e_i \cdot e_j$. It follows that, for any $Z \in V^{(2)}$, we have

$$Z = \sum_{(i,j)\in G_{2,n}} p(i,j) e^{\cdot}_{ij}, \quad p(i,j) \in R,$$

$\{p(i,j) | (i,j) \in G_{2,n}\}$ will be called coordinate of Z relative to the basis $\{e^{\cdot}_{ij} | (i,j) \in G_{2,n}\}$, where $p(i,j) = p(j,i)$.

An element $Z \in V^{(2)}$ is called decomposable if there exists $v_1, v_2 \in V$, such that $Z = v_1 \cdot v_2$. In this case, v_1 and v_2 are called the spanning vectors of the decomposable element Z. Obviously, $Z = \sum_{(i,j)\in G_{2,n}} p(i,j) e^{\cdot}_{ij}$ is decomposable if and only if there exists

$$v_1 = \sum_{j=1}^{n} a_{1j} e_j, \quad v_2 = \sum_{j=1}^{n} a_{2j} e_j.$$

such that

$$[1 + \delta(i,j)]^{-1} (a_{1i} a_{2j} + a_{2i} a_{1j}) = p(i,j) \tag{1}$$

*Received July 10, 1993.

for all $(i,j) \in G_{2,n}$, where $\delta(i,j)$ is Kronecker delta.

It is easy to verify that, for $Z \in V^{(2)}$, if $Z \neq 0$, then there exists $\{e_1, \cdots, e_n\}$, a proper basis of V, such that $p(1,1) \neq 0$, where $p(1,1)$ is coordinate of Z relative to the basis $\{e_{ij} | (i,j) \in G_{2,n}\}$. Hence, we shall always assume that $p(1,1) \neq 0$ in this paper.

Theorem *Let $Z = \sum_{(i,j)\in G_{2,n}} p(i,j)e_{ij}$, where $p(1,1) \neq 0$. Then Z is decomposable if and only if either*

i) $\sqrt{\Delta_j} \in R$, for all $j = 2, \cdots, n$, some one $\Delta_k \neq 0 (2 \leq k \leq n)$, and

$$\det \begin{bmatrix} 2p(1,1) & p(k,1) & p(1,j) \\ p(1,k) & 2p(k,k) & p(k,j) \\ p(1,i) & p(k,i) & [1+\delta(i,j)]p(i,j) \end{bmatrix} = 0 \tag{2}$$

for $i,j = 2, \cdots, k-1, k+1, \cdots, n$, or

ii) $\Delta_j = 0$, for all $j = 2, \cdots, n$, and

$$\det \begin{bmatrix} 2p(1,1) & p(1,j) \\ p(1,i) & [1+\delta(i,j)]p(i,j) \end{bmatrix} = 0 \tag{3}$$

for $i,j = 2, \cdots, n$, where $\Delta_j = p(1,j)^2 - 4p(1,1)p(j,j)$.

Proof Suppose $Z = \sum_{(i,j)\in G_{2,n}} p(i,j)e_{ij}$ is decomposable, and $p(1,1) \neq 0$. First, by the definition of spanning vectors of decomposable element, the spanning vectors of Z may be assumed as:

$$v_1 = \sum_{j=1}^{n} a_{1j}e_j, \quad v_2 = p(1,1)\sum_{j=1}^{n} a_{2j}e_j,$$

where $a_{11} = a_{21} = 1$.

In this time, (1) becomes

$$p(1,1)(a_{1i}a_{2j} + a_{2i}a_{1j}) = [1+\delta(i,j)]p(i,j). \tag{4}$$

Since that a_{1j}, a_{2j} are respectively two roots of the quatratic equation with one unknow[2]:

$$p(1,1)x^2 - p(1,j)x + p(j,j) = 0 \tag{5}$$

for $j = 2, \cdots, n$.

Then $\sqrt{\Delta_j} \in R, j = 2, \cdots, n$, where $\Delta_j = p(1,j)^2 - 4p(1,1)p(j,j)$.

Now, suppose $\Delta_k \neq 0 (2 \leq k \leq n)$. Similarly, by the definition of spanning vectors of decomposable element, we can take

$$a_{1k} = [2p(1,1)]^{-1}[p(1,k) + \sqrt{\Delta_k}], \quad a_{2k} = [2p(1,1)]^{-1}[p(1,k) - \sqrt{\Delta_k}]. \tag{6}$$

Next, taking $(i,j) = (1,j)$ and (k,j) in (4), $j \neq 1, k$, we obtain some systems of binary linear equations:

$$\begin{cases} a_{1j} + a_{2j} = p(1,1)^{-1}p(1,j) \\ a_{2k}a_{1j} + a_{1k}a_{2j} = p(1,1)^{-1}p(k,j) \end{cases} \quad j = 2, \cdots, k-1, k+1, \cdots, n.$$

Hence

$$a_{1j} = [p(1,1)(a_{1k} - a_{2k})]^{-1}[a_{1k}p(1,j) - p(k,j)],$$
$$a_{2j} = [p(1,1)(a_{1k} - a_{2k})]^{-1}[p(k,j) - a_{2k}p(1,j)]. \quad (7)$$

Substitute the dummy index i for j in (7), and then put the result and (7) into (4), we obtain

$$[1+\delta(i,j)]p(1,1)p(i,j)(a_{1k}-a_{2k})^2 = [p(1,i)p(k,j)+p(1,j)p(k,i)](a_{1k}+a_{2k}) - 2p(1,i)p(1,j)a_{1k}a_{2k} - 2p(k,i)p(k,j).$$

From (6), we can immediately obtain that

$$[1+\delta(i,j)][p(1,k)^2 - 4p(1,1)p(k,k)]p(i,j) = p(1,k)p(1,i)p(1,j) + p(1,k)p(k,i)p(1,j) - 2p(1,1)p(k,i)p(k,j) - 2p(k,k)p(1,i)p(1,j). \quad i,j = 2,\cdots,k-1,k+1,\cdots,n. \quad (8)$$

i.e., we have established the cubic homogenous expressions. Then we can immediately conclude from (8) that (2) holds for $i,j = 2,\cdots,k-1,k+1,\cdots,n$.

Finally, suppose $\Delta_j = 0$, for all $j = 2,\cdots,n$, then from (5), we get

$$a_{1j} = a_{2j} = [2p(1,1)]^{-1}p(1,j), \quad j = 2,\cdots,n.$$

Thus, (4) becomes

$$2[1+\delta(i,j)]p(1,1)p(i,j) = p(1,i)p(1,j), \quad (9)$$

i.e., we have established the quadratic homogenous expressions. Similary, we can immediately conclude from (9) that (3) holds $i,j = 2,\cdots,n$, the proof of the "only if" part is complete.

Assume now that $\sqrt{\Delta_j} \in R$, for all $j = 2,\cdots,n$, there exists a $\Delta_k \neq 0 (2 \leq k \leq n)$, and (2) (or (8)) holds, for $i,j = 2,\cdots,k-1,k+1,\cdots,n$. Set

$$v_1 = \sum_{j=1}^{n} a_{1j}e_j, \quad v_2 = p(1,1)\sum_{j=1}^{n} a_{2j}e_j$$

in which $a_{11} = a_{21} = 1, a_{1k}, a_{2k}$ are identical to (6), and a_{1j}, a_{2j} are identical to (7), $j = 2,\cdots,k-1,k+1,\cdots,n$.

Let $W = v_1 \cdot v_2 = \sum_{(i,j)\in G_{2,n}} q(i,j)e_{ij}$, then

$$q(i,j) = [1+\delta(i,j)]^{-1}p(1,1)(a_{1i}a_{2j} + a_{2i}a_{1j})$$

for $(i,j) \in G_{2,n}$.

First, it easy to see that

$$q(1,1) = p(1,1), \quad q(1,k) = p(1,k) \text{ and } q(k,k) = p(k,k).$$

Next, for $j = 2, \cdots, k-1, k+1, \cdots, n$, from (6) and (7), we have that

$$q(1,j) = p(1,1)(a_{1j} + a_{2j}) = p(1,j), \quad q(k,j) = p(1,1)(a_{1k}a_{2j} + a_{2k}a_{1j}) = p(k,j).$$

Moreover, for $i, j = 2, \cdots, k-1, k+1, \cdots, n$, we have that

$$q(i,j) = \frac{p(1,k)p(1,i)p(1,j) + p(1,k)p(k,i)p(1,j) - 2p(1,1)p(k,i)p(k,j) - 2p(k,k)p(1,i)p(1,j)}{[1+\delta(i,j)][p(1,k)^2 - 4p(1,1)p(k,k)]}$$

From (2) (or (8)), we get

$$q(i,j) = p(i,j) \quad \text{for} \quad i,j = 2, \cdots, k-1, k+1, \cdots, n.$$

Thus, we conclude that

$$q(i,j) = p(i,j)$$

for all $(i,j) \in G_{2,n}$. It implies that

$$Z = W = v_1 \cdot v_2,$$

i.e., Z is decomposable.

Now, if $\Delta_j = 0$, for all $j = 2, \cdots, n$, and (3) (or (9)) holds, for $i, j = 2, \cdots, n$, then set

$$v_1 = \sum_{j=1}^{n} a_{1j} e_j, \quad v_2 = p(1,1) \sum_{j=1}^{n} a_{2j} e_j,$$

where $a_{11} = a_{21} = 1, a_{1j} = a_{2j} = [2p(1,1)]^{-1} p(1,j)$, for $j = 2, \cdots, n$.

Similary, let $W = v_1 \cdot v_2 = \sum_{(i,j) \in G_{2,n}} q(i,j) e_{ij}$, then

$$q(i,j) = 2[1+\delta(i,j)]^{-1} p(1,1) a_{1i} a_{1j}, \quad (i,j) \in G_{2,n}.$$

Obviously, $q(1,1) = p(1,1)$ and $q(1,j) = p(1,j)$, for $j = 2, \cdots, n$.

Moreover, for $i,j = 2, \cdots, n$, we have $q(i,j) = [2p(1,1)(1+\delta(i,j))]^{-1} p(1,i)p(1,j)$. From (3) (or (9)), we get

$$q(i,j) = p(i,j), \quad \text{for} \quad i,j = 2, \cdots, n.$$

Hence, we can conclude that

$$Z = W = v_1 \cdot v_2,$$

i.e., Z is decomposable. The theorem has been proved.

Corollary Let v_1, v_2 be the spanning vectors of decomposable element

$$Z = \sum_{(i,j) \in G_{2,n}} p(i,j) e_{ij},$$

where $p(1,1) \neq 0$. If there exists a $\Delta_k \neq 0 (2 \leq k \leq n)$, then

$$v_1 = e_1 + \frac{p(1,k)+\sqrt{\Delta_k}}{2p(1,1)} e_k + \sum_{j \neq 1,k} \frac{p(1,j)\sqrt{\Delta_k} + p(1,k)p(1,j) - 2p(1,1)p(k,j)}{2p(1,1)\sqrt{\Delta_k}} e_j,$$

$$v_2 = p(1,1)e_1 + \frac{p(1,k)-\sqrt{\Delta_k}}{2} e_k + \sum_{j \neq 1,k} \frac{p(1,j)\sqrt{\Delta_k} - p(1,k)p(1,j) + 2p(1,1)p(k,j)}{2\sqrt{\Delta_k}} e_j,$$

otherwise,

$$v_1 = e_1 + \sum_{j=2}^{n} \frac{p(1,j)}{2p(1,1)} e_j, \quad v_2 = p(1,1)e_1 + \sum_{j=2}^{n} \frac{p(1,j)}{2} e_j,$$

where $\Delta_k = p(1,k)^2 - 4p(1,1)p(k,k)$.

References

[1] Wang Boying, *The Basic of Multilinear Algebra*, Beijing Normal University Press, 1985.

[2] Zhu Zhongnan, *A necessary and sufficient condition satisfied by decomposable elements in the $V^{(2)}$ completely symmetric tensor space*, Journal of Nanjing University (Natural Sciences Edition), Vol.25(1989), 705–710.

[3] R.Merris, *Equality of decomposable symmetric tensors* , Canal. J. Math., **27**(1975), 1022–1027.

[4] J.A.Dias da Silva, *Condition for equality of decomposable symmetric tensors* , Linear Algebra and Appl., **24**(1979), 85–92.

二阶完全对称张量空间中可分元素的坐标间的齐次关系

朱 忠 南

(江苏教育学院数学系，南京 210013)

摘　要

本文给出了二阶完全对称张量空间$V^{(2)}$中可分元素的坐标间的齐次关系，特别地，这种齐次关系可以由行列式表示出来．并由此得到了$V^{(2)}$中可分元素的一个等价条件．