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The Evaluation of Basic Hypergeometric Series (I) *

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Abstract A simple algorithm for the evaluation of basic hypergeometric series is established, as a consequence, some interesting summation formulas are obtained.

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Let a, q be complex numbers, $0 < |q| < 1$, we denote

$$(a;q)_n = (a)_n = \prod_{i=1}^{n}(1 - aq^{i-1}), (a)_0 = 1, (a)_\infty = \prod_{i=1}^{\infty}(1 - aq^{i-1}).$$

In the notation above, a basic hypergeometric series ${}_r\phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} ; q, z \right]$ may be defined by (see [1], p4)

$$ {}_r\phi_s \left[\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{i=1}^{s}(b_i)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} \frac{z^n}{(q)_n}, \qquad (1)$$

where $r = 1 + s, |z| < 1; r \leq s, |z| < +\infty$.

Moreover, we define

$$\psi_q(x) = \frac{d}{dx} \ln |\Gamma_q(x)|, \qquad (2)$$

for $0 < q < 1$, where $\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1 - q)^{1-x}$ is a q-gamma function (see [1], p16).

Obviously, we have

$$\psi_q(x) = -\ln(1 - q) + \sum_{n=0}^{\infty} \frac{q^{x+n} \ln q}{1 - q^{x+n}}, \qquad (3)$$

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and

$$\psi_q(x) \to -\ln(1-q), \quad x \to +\infty. \tag{4}$$

From definition of series (1), it can be obtained that

$$\frac{(-1)^{1+s-r}z}{(1-q)}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1q, a_2q, \cdots, a_rq, q \\ b_1q, b_2q, \cdots, b_sq, q^2 \end{matrix}; q, q^{1+s-r}z\right]$$
$$= \frac{\prod_{i=1}^{s}(1-b_i)}{\prod_{i=1}^{r}(1-a_i)}\left\{{}_r\phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r, \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right] - 1\right\} \tag{5}$$

In case that the basic hypergeometric function ${}_r\phi_s$ on the right-hand side of (5) can be evaluated in a closed form, the identity (5) will yield a summation formula for the (hight-order) basic hypergeometric function ${}_{r+1}\phi_{s+1}$ occuring on the left-hand side. Furthermore, it follows from (5) that

$$\frac{(-1)^{1+s-r}z}{(1-q)}{}_{r+1}\phi_{s+1}\left[\begin{matrix} a_1q, a_2q, \cdots, a_rq, q \\ b_1q, b_2q, \cdots, b_sq, q^2 \end{matrix}; q, q^{1+s-r}z\right]\Bigg|_{a_k=1}$$
$$= \lim_{a_k\to 1}\left\{\frac{\prod_{i=1}^{s}(1-b_i)}{\prod_{i=1}^{r}(1-a_i}\left({}_r\phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r, \\ b_1, b_2, \cdots, b_s \end{matrix}; q, z\right] - 1\right)\right\} \tag{6}$$

for each integer $k(1 \le k \le r)$, we shall make use of (6) to give the sum of some basic hypergeometric series.

Let $0 < q < 1$, then we have

Theorem 1

$${}_9\phi_8\left[\begin{matrix} aq, q^2\sqrt{a}, -q^2\sqrt{a}, bq, cq, \frac{a^2q^{2+n}}{bc}, q^{1-n}, q, q \\ q\sqrt{a}, -q\sqrt{a}, aq^2/b, aq^2/c, bc/(aq^{n-1}), aq^{2+n}, aq^2, q^2 \end{matrix}; q, q\right]$$
$$= \frac{(1-q)(1-aq/b)(1-aq/c)(1-bc/(aq^n))(1-aq)(1-aq^{n+1})}{q(1-aq^2)(1-b)(1-c)(1-a^2q^{1+n}/(bc))(1-q^{-n})\ln q}$$
$$\times\{\psi_q(1+\frac{\ln a}{\ln q}) - \psi_q(n+1+\frac{\ln a}{\ln q}) + \psi_q(1+\frac{\ln a - \ln bc}{\ln q}) - \psi_q(n+1+\frac{\ln a - \ln bc}{\ln q})$$
$$+\psi_q(n+1+\frac{\ln a - \ln b}{\ln q}) - \psi_q(1+\frac{\ln a - \ln b}{\ln a}) + \psi_q(n+1+\frac{\ln a - \ln c}{\ln q})$$
$$-\psi_q(1+\frac{\ln a - \ln c}{\ln q})\}.$$

Corollary 1

$${}_7\phi_6\left[\begin{matrix} aq, q^2\sqrt{a}, -q^2\sqrt{a}, bq, cq, q, q \\ q\sqrt{a}, -q\sqrt{a}, aq^2/b, aq^2/c, aq^2, q^2 \end{matrix}; q, \frac{aq}{bc}\right]$$
$$= \frac{bc(1-q)(1-aq/b)(1-aq/c)(1-aq)}{aq(1-aq^2)(1-b)(1-c)\ln q}\{\psi_q(1+\frac{\ln a}{\ln q}) + \psi_q(1+\frac{\ln a - \ln bc}{\ln q})$$
$$-\psi_q(1+\frac{\ln a - \ln b}{\ln q}) - \psi_q(1+\frac{\ln a - \ln c}{\ln q})\}.$$

Corollary 2

$$
{}_7\phi_6\left[\begin{matrix} aq, q^2\sqrt{a}, -q^2\sqrt{a}, bq, q^{1-n}, q, q \\ q\sqrt{a}, -q\sqrt{a}, aq^2/b, aq^{n+2}, aq^2, q^2 \end{matrix}; q, \frac{aq^{1+n}}{b}\right]
$$
$$
= \frac{b(1-q)(1-aq)(1-aq/b)(1-aq^{n+1})}{aq^{1+n}(1-b)(1-aq^2)(1-q^{-n})\ln q}\{\psi_q(1+\frac{\ln a}{\ln q}) - \psi_q(n+1+\frac{\ln a}{\ln q})
$$
$$
+\psi_q(n+1+\frac{\ln a - \ln b}{\ln q}) - \psi_q(1+\frac{\ln a - \ln b}{\ln q})\}.
$$

Theorem 2

$$
{}_{r+3}\phi_{r+2}\left[\begin{matrix} aq, b_1q^{m_1+1}, \cdots, b_rq^{m_r+1}, q, q \\ b_1q, b_2q, \cdots, b_rq, q^2, q^2 \end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right]
$$
$$
= -\frac{(1-q)^2\prod_{i=1}^r(1-b_i)}{a^{-1}q^{1-(m_1+\cdots+\cdots+m_r)}(1-a)\prod_{i=1}^r(1-b_iq^{m_i})}\{\frac{1}{\ln q}(\sum_{i=1}^r(\psi_q(\frac{\ln b_i}{\ln q})
$$
$$
-\psi_q(m_i+\frac{\ln b_i}{\ln q})) + \psi_q(1) - \psi_q(1-\frac{\ln a}{\ln q})) + (m_1+m_2+\cdots+m_r)\},
$$

where $m_1, m_2, \cdots, m_r$ are arbitrary nonegative integers.

Theorem 3

$$
{}_{r+2}\phi_{r+1}\left[\begin{matrix} aq, b_1q^{m_1+1}, \cdots, b_rq^{m_r+1}, q \\ b_1q, b_2q, \cdots, b_rq, q^2 \end{matrix}; q, a^{-1}q^{-(m_1+\cdots+m_r)}\right]
$$
$$
= -\frac{aq^{m_1+m_2+\cdots+m_r}(1-q)\prod_{i=1}^r(1-b_i)}{(1-a)\prod_{i=1}^r(1-b_iq^{m_i})},
$$

where $m_1, m_2, \cdots, m_r$ are arbitrary nonnegative integers.

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