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On the Construction of Approximate Inertial Manifolds *

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An approximate inertial manifold is a smooth finite manifold such that every orbit enters its a small neighborhood after a finite time. In this paper, we construct an approximate inertial manifold for the following reaction diffusion equations:

$$\frac{\partial u}{\partial t} - d\Delta u + g(u) = 0 \quad \text{in} \quad R^+ \times \Omega, \tag{1}$$

with

$$u(x,0) = u_0(x) \quad \text{and} \quad u|_{\partial\Omega} = 0, \tag{2}$$

where $d > 0$ and Ω is a bounded regular subset of $R^n (n \leq 4)$. g is C^2 function from R to R which satisfies

$$g'(s) \geq -c_1 \quad \text{and} \quad c_2|s|^k - c_4 \leq g(s)s \leq c_3|s|^k + c_4 \tag{3}$$

for $s \in R$ with $k > 2, c_i > 0$.

Let $Au = -d\Delta u + u$, then A is an unbounded positive self-adjoint operator on $H = L^2(\Omega)$ with domain $D(A) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\}$. Since A^{-1} is compact, there exists an orthonormal basis of H consisting of eigenvectors w_j of A, i.e., $Aw_j = \lambda_j w_j (j = 1, \cdots)$. Under assumptions (3), it follows from [1] that for $u_0 \in H$, the problem (1) and (2) have a unique solution $u(t)$ defined on R^+ such that $\|u(t)\| \leq M_0, |u(t)|_\infty \leq M_0$ for $t \geq t_0$, where t_0 depends on u_0 and M_0 is a constant. Here and after, we denote by $\|\cdot\|$ the norm of $D(A^{\frac{1}{2}})$. Let $\phi : R^+ \to R$ be a smooth truncation function such that $\phi(s) = 1$, if $0 \leq s \leq 1; \phi(s) = 0$ if $s \geq 2$. Set $f(s) = \phi(\frac{s^2}{M_0^2})(g(s) - s)$ for $s \in R$, then when $t \geq t_0, u(t)$ satisfies

$$\frac{du}{dt} + Au + f(u) = 0, \tag{4}$$

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for m given, denote by $P_m : H \to \text{span}(w_1, \cdots, w_m)$ the projector and $Q_m = I - P_m$. We introduce

$$B_m = \{y \in P_m H : \|y\| \le 2M_0\}, \quad B_m^* = \{z \in Q_m H : \|z\| \le 2M_0\}.$$

For $y \in B_m$, consider the following implicit system

$$y_1 + Ay + P_m f(y+z) = 0, \tag{5}$$

$$Az_1 + Q_m f'(y+z)(y_1 + z_1) = 0, \tag{6}$$

$$z_1 + Az + Q_m f(y+z) = 0. \tag{7}$$

By a fixed point argument, we may show

Theorem 1 *There exists an integer m_0 such that when $m \ge m_0$, for all $y \in B_m$ the system (5)–(6) has a unique solution $y_1(y) \in P_m H, z_1(y) \in Q_m H$, and $z(y) \in B_m^*$.*

By Theorem 1, we can define a mapping $\xi : B_m \to B_m^*$ such that for $y \in B_m, \xi(y) = z(y)$. Let $\Sigma =$ graph (ξ), then our main results are obtained:

Theorem 2 *Assume that (3) holds. Then there exists m_1 such that when $m \ge m_1, \Sigma$ is an approximate inertial manifold of (1)–(2), and any solution $u(t)$ of (1)–(2) remains at a distance of Σ in H bounded by $M_1(\frac{\lambda_1}{\lambda_{m+1}})^3$ for all $t \ge t_1$, where t_1 depending on u_0 and M_1 is a constant.*

We remark that because of $\lambda_m \to \infty$ as $m \to \infty$, Theorem 2 implies that the distance of $u(t)$ from Σ can be made arbitrarily small by choosing m large enough.

References

[1] M.Marion, *Approximate inertial manifolds for reaction diffusion equations in high space dimension*, J. Dynamics Differential Equations, Vol.1, No.3(1989), 245–267.