关于 L_1 空间多项式导数的估计*

孙燮华
(中国计量学院，杭州 310034)

摘 要 本文在加权 L_1 空间中对正导数多项式建立了精确的 Bernstein 型不等式.

关键词 L_1 空间，Bernstein 型不等式.
分类号 AMS(1991) 41A17，26D05/CCL O141.41

多项式导数的估计在逼近论的逆定理的建立中起着重要的作用(见[1])，同时建立多项式导数的估计不等式，即所谓 Bernstein 型或 Markoff 型不等式，也是一个有意义的独立课题. 本短文的目的是在加权 L_1 空间中对正系数多项式的导数建立精确的估计不等式. 我们的主要结果是

定理 设 $p_n(x)$ 是 n 次正系数多项式，则

(i) 当 $\alpha\geqslant 0$ 时，

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx\leqslant\frac{n}{n+\alpha}\int_0^\infty p_n(x)x^\alpha e^{-x}dx, \tag{1}$$

上述不等式的等号对于多项式 $p_n(x)=x^n$ 成立.

(ii) 当 $-1<\alpha<0$ 时，

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx\leqslant\frac{1}{1+\alpha}\int_0^\infty p_n(x)x^\alpha e^{-x}dx, \tag{2}$$

这里系数 $(1+\alpha)^{-1}$ 不能再减小.

证明 设 $p_n(x)$ 是任一 n 次正系数多项式，写

$$p_n(x)=a_nx^n+p_{n-1}(x),$$

这里

$$p_{n-1}(x)=\sum_{k=0}^{n-1}a_kx^k,\quad a_k\geqslant 0(k=0,1,\cdots,n).$$

于是，

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx=na_n\Gamma(n+\alpha)+\int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx \tag{3}$$

和

$$\int_0^\infty p_n(x)x^\alpha e^{-x}dx=a_n(n+\alpha)\Gamma(n+\alpha)+\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx. \tag{4}$$

* 1992年5月26日收到.

记

$$b_n = n/(n+\alpha) \quad (n=1,2,\cdots).$$

显然

$$b_n \geqslant b_{n-1}, \quad 当 \alpha \geqslant 0, \\ b_n < b_{n-1}, \quad 当 -1<\alpha<0. \tag{5}$$

若 $\alpha \geqslant 0$,由(3)和(4),用(3.3)得

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx$$
$$\leqslant \int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_{n-1}\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx.$$

多次应用上述不等式,得

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_{n-1}\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx$$
$$\leqslant \cdots \leqslant \int_0^\infty p_1'(x)x^\alpha e^{-x}dx - b_1\int_0^\infty p_1(x)x^\alpha e^{-x}dx, \tag{6}$$

这里 $p_1(x)=a_1x+a_0, a_1\geqslant 0, a_0\geqslant 0$. 直接计算,得

$$\int_0^\infty p_1'(x)x^\alpha e^{-x}dx - b_1\int_0^\infty p_1(x)x^\alpha e^{-x}dx = a_1\Gamma(\alpha+1) - \frac{1}{1+\alpha}(a_1\Gamma(2+\alpha)+a_0\Gamma(\alpha+1))$$
$$= -\frac{a_0}{1+\alpha}\Gamma(\alpha+1) \leqslant 0. \tag{7}$$

由(6)和(7)推出(1). 至于(1)的等号成立,可对 $p_n(x)=x^n$ 直接验证.

若 $-1<\alpha<0$,由(3)和(4)

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x} = \int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_{n-1}\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx$$
$$+ (b_{n-1}-b_n)\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx. \tag{8}$$

因为对于 $1\leqslant m\leqslant n$,

$$\int_0^\infty p_m(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_n(x)x^\alpha e^{-x}dx, \tag{9}$$

此处 $p_m(x) = \sum_{k=0}^m a_kx^k, a_k(k=0,1,\cdots,m)$ 是多项式 $p_n(x)$的系数,所以

$$(b_{n-1}-b_n)\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx \leqslant (b_{n-1}-b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx.$$

将上式代入(8),有

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_{n-1}\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx$$
$$+ (b_{n-1}-b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx. \tag{10}$$

在(10)中,用$(n-1)$代替 n 得

$$\int_0^\infty p_{n-1}'(x)x^\alpha e^{-x}dx - b_{n-1}\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_{n-2}'(x)x^\alpha e^{-x}dx - b_{n-2}\int_0^\infty p_{n-2}(x)x^\alpha e^{-x}dx$$
$$+ (b_{n-2}-b_{n-1})\int_0^\infty p_{n-1}(x)x^\alpha e^{-x}dx. \tag{11}$$

结合(10)和(11)并再次用(9)得

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_{n-2}'(x)x^\alpha e^{-x}dx - b_{n-2}\int_0^\infty p_{n-2}(x)x^\alpha e^{-x}dx$$

$$+ (b_{n-1} - b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx + (b_{n-2} - b_{n-1})\int_0^\infty p_n(x)x^\alpha e^{-x}dx$$

$$= \int_0^\infty p_{n-2}'(x)x^\alpha e^{-x}dx - b_{n-2}\int_0^\infty p_{n-2}(x)x^\alpha e^{-x}dx + (b_{n-2} - b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx.$$

重复应用(9)和(10)推出

$$\int_0^\infty p_n'(x)x^\alpha e^{-x}dx - b_n\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant \int_0^\infty p_1'(x)x^\alpha e^{-x}dx - b_1\int_0^\infty p_1(x)x^\alpha e^{-x}dx$$

$$+ (b_1 - b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx \leqslant (b_1 - b_n)\int_0^\infty p_n(x)x^\alpha e^{-x}dx. \qquad (12)$$

上面最后一个不等式的推导中,利用了(7). 最后,(2)直接由(10)推出.

现设

$$q_n(x) = x^n + \lambda x, \quad \lambda > 0.$$

显然

$$\frac{\int_0^\infty q_n'(x)x^\alpha e^{-x}dx}{\int_0^\infty q_n(x)x^\alpha e^{-x}dx} = \frac{n\Gamma(n+\alpha) + \lambda\Gamma(\alpha+1)}{\Gamma(n+\alpha+1) + \lambda\Gamma(\alpha+2)}$$

$$\to \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)} = \frac{1}{\alpha+1} \quad (\lambda \to \infty).$$

由此可知,(2)中的系数$(\alpha+1)^{-1}$不能再减小.

参 考 文 献

[1] G. G. Lorentz, *Approximation of Functions*, Holt, Rinehart & Winston, N. Y., 1966.

On the Estimation of Derivatives of Polynomials in L_1 Space

Sun Xiehua

(China Institute of Metrology, Hangzhou 310034)

Abstract

In weighted L_1 space, two exact inequalities for derivatives of polynomials with positive coefficients are established.

Keywords L_1 space, Bernstein type inequality.