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正态——正态结构串联系统可靠度的Bayes估计*

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摘　要　本文利用Bayes方法讨论了正态——正态结构串联系统可靠度 $\theta=P_r(X>Y)$ 的估计问题，把 θ 的后验均值作为它的点估计，在特殊情况下，还得到了 θ 的Bayes置信限.

关键词　可靠度，Bayes置信限.

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一个部件或系统称为失效，如果当其所承受的应力超过它本身的强度时.而部件或系统的强度 X 及应力 Y 往往是未知的随机变量，因此估计部件或系统的可靠度 $\theta=P_r(X>Y)$ 是很重要的.自从Birnbaum[1](1956)发表了利用 U 统计量求得的 θ 的MVUE以来，人们在可靠度 θ 的点估计方面作了大量工作.本文利用Bayes方法得到了 θ 的点估计和 θ 的Bayes置信限.

一　部件强度相互独立的串联系统承受同一应力时可靠度的估计

设系统是串联的，$X_1,X_2,\cdots,X_p$ 为部件的强度，Y 为系统所受的应力，$X_1,X_2,\cdots,X_p,Y$ 相互独立且 $X_i\sim N(\mu_i,\sigma^2)$，$i=1,2,\cdots,p$，$Y\sim N(\mu,\sigma^2)$，此时系统的可靠度

$$\theta=P_r(X_1>Y,\cdots,X_p>Y)=P_r(X_1-Y>0,\cdots,X_p-Y>0)$$
$$=(2\pi)^{-\frac{p}{2}}\sigma^{-p}|\Lambda|^{-\frac{1}{2}}\int\underset{\underset{\sim}{z}<0}{\cdots}\int\exp\{-\frac{1}{2}(\underset{\sim}{z}-\underset{\sim}{v})^{\tau}\frac{\Lambda^{-1}}{\sigma^2}(\underset{\sim}{Z}-\underset{\sim}{v})\}d\underset{\sim}{z},$$

其中 $\underset{\sim}{v}=(v_1,v_2,\cdots,v_p)^{\tau}$，$v_i=\mu-\mu_i$，$i=1,2,\cdots,p$

$$\underset{\sim}{Z}=(z_1,z_2,\cdots,z_p)^{\tau},\ \Lambda=\Lambda_{p\times p}=\begin{pmatrix}2&1&\cdots&1\\1&2&\cdots&1\\\cdots&\cdots&\cdots&\cdots\\1&1&\cdots&2\end{pmatrix}$$

设 $X_{ij}=1,2,\cdots,n_i$ 为 X_i 的观测值，$i=1,2,\cdots,p$，$Y_1,Y_2,\cdots,Y_{n_{p+1}}$ 为 Y 的观测值.记

$$N=\sum_{i=1}^{p+1}n_i,\ \bar{X}_i=\frac{1}{n_i}\sum_{j=1}^{n_i}X_{ij},\ i=1,\cdots,p,\ \bar{Y}=\frac{1}{n_{p+1}}\sum_{j=1}^{n_{p+1}}Y_j,$$

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$$S^2=(N-P-1)^{-1}[\sum_{i=1}^{p}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2+\sum_{j=1}^{n_{p+1}}(Y_i-\bar{Y})^2],$$

则$(\mu_1,\mu_2,\cdots,\mu_p,\mu,\sigma)$的似然函数是

$$L(\mu_1,\mu_2,\cdots,\mu_p,\mu,\sigma)$$
$$=\sigma^{-N}\exp\{-\frac{1}{2}\sigma^{-2}[(N-P-1)S^2+\sum_{i=1}^{p}n_i(\bar{X}_i-\mu_i)^2+n_{p+1}(\bar{Y}-\mu)^2]\}.$$

假定$(\mu_1,\mu_2,\cdots,\mu_p,\mu,\sigma)$先验分布为“共轭型”，且在$\sigma$给定条件下，$\mu_1,\mu_2,\cdots,\mu_p,\mu$相互独立，即

$$P(\mu_1,\mu_2,\cdots,\mu_p,\mu,\sigma)\propto\sigma^{-(p+r+2)}\exp\{-\frac{1}{2}\sigma^{-2}[r\beta+(\underset{\sim}{w}-\underset{\sim}{m})^\tau\Lambda(\underset{\sim}{w}-\underset{\sim}{m})]\},$$

其中 $\underset{\sim}{w}=(\mu_1,\cdots,\mu_p,\mu)^\tau$，$\underset{\sim}{m}=(m_1,m_2,\cdots,m_{p+1})^\tau$，$r>0$，$\beta>0$

$$\Lambda=\Lambda_{(p+1)\times(p+1)}=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_{p+1}\end{pmatrix},\ \lambda_i>0, i=1,\cdots,p+1.$$

由 Bayes 定理，$(\mu_1,\cdots,\mu_p,\mu,\sigma)$的后验密度函数为

$$P(\mu_1,\cdots,\mu_p,\mu,\sigma\mid\bar{X}_1\bar{X}_2\cdots\bar{X}_p,\bar{Y},S^2)\propto\sigma^{-(N+p+r+2)}$$
$$\cdot\exp\{-\frac{1}{2\sigma^2}[(N-p-1)S^2+r\beta+(\bar{X}-\underset{\sim}{m})^\tau\Delta(\Delta+\Lambda)^{-1}\Lambda(\bar{X}-\underset{\sim}{m})]$$
$$-\frac{1}{2\sigma^2}[\underset{\sim}{w}-(\Delta+\Lambda)^{-1}(\Delta\bar{X}+\Lambda\underset{\sim}{m})]^\tau(\Delta+\Lambda)[\underset{\sim}{w}-(\Delta+\Lambda)^{-1}(\Delta\bar{X}+\Lambda\underset{\sim}{m})]\}$$

这里 $\bar{X}=(\bar{X}_1,\bar{X}_2,\cdots,\bar{Y}_p,\bar{Y})^\tau$，$\Delta=\begin{pmatrix}n_1 & & 0\\ & \ddots & \\ 0 & & n_{p+1}\end{pmatrix}$

从而在σ给定条件下

$$\mu_i\sim N(\frac{n_i\bar{X}_i+\lambda_i m_i}{n_i+\lambda_i},\frac{\sigma^2}{n_i+\lambda_i}),\ i=1,2,\cdots,p,$$

$$\mu\sim N(\frac{\lambda_{p+1}m_{p+1}+n_{p+1}\bar{Y}}{n_{p+1}+\lambda_{p+1}},\frac{\sigma^2}{n_{p+1}+\lambda_{p+1}}).$$

于是

$$Ev_i=E[\mu-\mu_i]=(\lambda_{p+1}m_{p+1}+n_{p+1}\bar{Y})(n_{p+1}+\lambda_{p+1})^{-1}-(\lambda_i m_i+n_i\bar{X}_i)(\lambda_i+n_i)^{-1}\stackrel{\triangle}{=}a_i,$$

$$E[\underset{\sim}{v}]=\frac{\lambda_{p+1}m_{p+1}+n_{p+1}\bar{Y}}{\lambda_{p+1}+n_{p+1}}\underset{\sim}{1}-(\Delta_1+\Lambda_1)^{-1}(\Lambda_1\underset{\sim}{m}_1+\Delta_1\bar{X}^{(1)})\stackrel{\triangle}{=}\underset{\sim}{a},\quad i=1,2,\cdots,p$$

其中

$$\underset{\sim}{1}=(1,1,\cdots,1)^\tau_{1\times p},\underset{\sim}{m}_1=(m_1,m_2,\cdots,m_p)^\tau,$$

$$\bar{X}^{(1)}=(\bar{X}_1,\bar{X}_2,\cdots,\bar{X}_p)^\tau,\underset{\sim}{\mu}=(\mu_1,\cdots,\mu_p)^\tau,$$

$$\underset{\sim}{v}=(v_1,\cdots,v_p)^\tau,\underset{\sim}{a}=(a_1,\cdots,a_p)^\tau,$$

$$\Lambda_1=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_p\end{pmatrix},\ \Delta_1=\begin{pmatrix}n_1 & & 0\\ & \ddots & \\ 0 & & n_p\end{pmatrix}.$$

所以

$$\frac{1}{\sigma^2}\mathrm{cov}(\underset{\sim}{v},\underset{\sim}{v})=(\Delta_1+\Lambda_1)^{-1}+(\lambda_{p+1}+n_{p+1})^{-1}\underset{\sim}{1}\underset{\sim}{1}^{\tau}\overset{\triangle}{=}B,$$

$$P(\underset{\sim}{v}/\sigma,\overline{X}_1,\cdots,\overline{X}_p,\overline{Y},S^2)=(2\pi)^{-\frac{p}{2}}\sigma^{-p}|B|^{-\frac{1}{2}}\exp\{-\frac{1}{2\sigma^2}(\underset{\sim}{v}-\underset{\sim}{a})^{\tau}B^{-1}(\underset{\sim}{v}-\underset{\sim}{a})\}.$$

又由 σ^2 的后验分布密度

$$P(\sigma/\overline{X}_1,\cdots,\overline{X}_p,\overline{Y},S^2)=\frac{2(\frac{b}{2})^{\frac{N+r}{2}}\cdot\exp\{-\frac{\sigma^2}{2}b\}}{\sigma^{(N+r+1)}\Gamma(\frac{N+r}{2})}I_{(\sigma>0)},$$

$$(b=(N-p-1)S^2+r\beta+(\underset{\sim}{m}-\overline{X})^{\tau}\Delta(\Delta+\Lambda)^{-1}\Lambda(\underset{\sim}{m}-\overline{X})).$$

$(v_1,\cdots,v_p,\sigma)$的后验分布密度函数为

$$P\{\underset{\sim}{v},\sigma/\overline{X}_1,\cdots,\overline{X}_p,\overline{Y},S^2\}=\frac{2(\frac{b}{2})^{\frac{N+r}{2}}\exp\{-\frac{1}{2\sigma^2}[b+(\underset{\sim}{v}-\underset{\sim}{a})^{\tau}B^{-1}(\underset{\sim}{v}-\underset{\sim}{a})]\}}{(2\pi)^{\frac{p}{2}}|B|^{\frac{1}{2}}\Gamma(\frac{N+r}{2})\sigma^{N+p+r+1}}I_{(\sigma>0)}.$$

因此得 θ 的 Bayes 估计量

$$E[\theta/\overline{X}_1,\cdots,\overline{X}_p,\overline{Y},S^2]=\frac{b^{\frac{1}{2}(N+r)}|\Lambda|^{-\frac{1}{2}}|B|^{-\frac{1}{2}}}{2^{\frac{1}{2}(N+r+2p-2)}\pi^p\Gamma(\frac{N+r}{2})}\int\cdots\int_{\underset{\sim}{z}<0}\int_{\sigma>0}\int\cdots\int_{\underset{\sim}{v}}\sigma^{-(N+2p+r+1)}$$

$$\cdot\exp\{-\frac{1}{2\sigma^2}[(\underset{\sim}{v}-\underset{\sim}{z})^{\tau}\Lambda^{-1}(\underset{\sim}{v}-\underset{\sim}{z})+b+(\underset{\sim}{v}-\underset{\sim}{a})^{\tau}B^{-1}(\underset{\sim}{v}-\underset{\sim}{a})]\}d\underset{\sim}{v}d\sigma d\underset{\sim}{z}$$

$$=\frac{\Gamma(\frac{N+p+r}{2})b^{-\frac{p}{2}}}{\pi^{\frac{p}{2}}|\Lambda+B|\Gamma(\frac{N+r}{2})}\int\cdots\int_{\underset{\sim}{z}<\underset{\sim}{a}}[1+\frac{\underset{\sim}{z}^{\tau}(\Lambda+B)^{-1}\underset{\sim}{z}}{b}]^{-\frac{1}{2}(N+p+r)}d\underset{\sim}{z}$$

注意到 $\lim\limits_{N\to\infty}\frac{b}{N}=\sigma^2$，$\lim\limits_{\substack{n_i\to\infty\\ 1\leqslant i\leqslant p+1}}(\Lambda+B)=\Lambda(\mathrm{a.s})$ 知 $E[\theta|\overline{X}_1,\cdots,\overline{X}_p,\overline{Y},S^2]$为 θ 的强相合估计.

二　部件强度独立同分布的串联系统承受同一应力时可靠度的估计

设系统的 p 个串联部件的强度 $X_1,\cdots,X_p$ i. i. d $\sim N(\mu,\sigma^2)$，而系统的应力 $Y\sim N(v,\sigma^2)$，设 $X_1,\cdots,X_p$ 与 Y 相互独立，则系统可靠度

$$\theta=P_r(X_1>Y,\cdots,X_p>Y)=(2\pi)^{-\frac{p}{2}}|\Lambda|^{-\frac{1}{2}}\int\cdots\int_{\underset{\sim}{z}<\frac{m}{\sigma}\underset{\sim}{1}}\exp\{-\frac{1}{2}\underset{\sim}{z}\Lambda^{-1}\underset{\sim}{z}\}d\underset{\sim}{z},$$

其中 $\underset{\sim}{1}=(1,\cdots,1)^{\tau}_{1\times p}$, $\underset{\sim}{z}=(z_1,\cdots,z_p)^{\tau}$, $m=v-\mu$, $\Lambda=\begin{pmatrix}2&1&\cdots&1\\1&2&\cdots&1\\\cdots&\cdots&\cdots&\\1&1&\cdots&2\end{pmatrix}_{p\times p}$,

若 $X_1,\cdots,X_N$ 为 $N(\mu,\sigma^2)$ 的样本, $Y_1,\cdots,Y_M$ 为 $N(v,\sigma^2)$ 的样本, $\overline{X}=\frac{1}{N}\sum_{j=1}^{N}X_i$, $\overline{Y}=\frac{1}{M}\sum_{j=1}^{M}Y_i$, $S^2=(M+N-2)^{-1}[\sum_{i=1}^{N}(X_i-\overline{X})^2+\sum_{i=1}^{M}(Y_i-\overline{Y})^2]$, (μ,v,σ) 的联合先验密度为"无信息先验" $p(\mu,v,\sigma)\infty\frac{1}{\sigma}I_{(\sigma<0)}$, 则 θ 的后验均值

$$E[\theta|\overline{X},\overline{Y},S^2]=\frac{(MN)^{\frac{1}{2}}[(M+N-2)S^2]^{-\frac{1}{2}}\Gamma(\frac{N+M+P-2}{2})}{\pi^{p/2}[P(M+N+MN)+MN]^{p/2}\Gamma(\frac{M+N-2}{2})}$$

$$\cdot\int_{\underset{\sim}{z}<T-\overline{X}}\cdots\int\{1+[(M+N-2)S^2]^{-1}\underset{\sim}{z}^{\tau}[I_p-\frac{(MN+M+N)\underset{\sim}{1}\underset{\sim}{1}^{\tau}}{P(M+N+MN)+MN}]\underset{\sim}{z}\}^{-\frac{1}{2}(M+N+P-2)}d\underset{\sim}{z}.$$

由 θ 为 $\frac{m}{\sigma}$ 的单调函数, 令 $t=\frac{m}{\sigma}$, 则 t 的后验密度函数是

$$P(t|\overline{X},\overline{Y},S^2)=\frac{(MN)^{\frac{1}{2}}[(M+N-2)S^2]^{\frac{1}{2}(M+N-2)}\exp\{-\frac{1}{2}\frac{MN}{M+N}t^2\}}{(2\pi)^{\frac{1}{2}}(M+N)^{\frac{1}{2}}[(M+N-2)S^2+\frac{MN}{M+N}a^2]^{\frac{1}{2}(M+N-2)}\Gamma(\frac{M+N-2}{2})}$$

$$\cdot\sum_{j=0}^{\infty}\frac{\Gamma(\frac{M+N+j-2}{2})}{\Gamma(j+1)}\left(\frac{\sqrt{2}\frac{MN}{M+N}at}{[(M+N-2)S^2+\frac{MN}{M+N}a^2]^{\frac{1}{2}}}\right)^j,$$

其中 $a=\overline{Y}-\overline{X}$. 所以

$$E(t)=\frac{\sqrt{2}a}{[(M+N-2)S^2]^{\frac{1}{2}}}\cdot\frac{\Gamma(\frac{M+N-1}{2})}{\Gamma(\frac{M+N-2}{2})},$$

$$D(t)=\frac{(M+N-2)\Gamma^2[\frac{1}{2}(M+N-2)]-2\Gamma^2[\frac{1}{2}(M+N-1)]}{[(M+N-2)S^2]\Gamma^2[\frac{1}{2}(M+N-2)]}a^2+\frac{M+N}{MN}.$$

令 $\rho=\frac{t-E(t)}{\sqrt{D(t)}}$, 由[3]得 ρ 的特征函数

$$E[\exp(i\rho(\beta))]=\exp\{-\frac{\beta^2}{2}\frac{M+N}{MN}\cdot\frac{1}{D(t)}-i\beta E(t)\cdot\frac{1}{D(t)}\}$$

$$\cdot\sum_{j=1}^{\infty}\frac{(\sqrt{2}i\beta)^j}{\Gamma(j+1)}\left[\frac{a[(M+N-2)S^2]^{-\frac{1}{2}}}{\sqrt{D(t)}}\right]^j\frac{\Gamma[\frac{1}{2}(M+N+j-2)]}{\Gamma[\frac{1}{2}(M+N-2)]}$$

利用 Stirling 公式可得 $E[\exp(i\rho\beta)]\simeq\exp\{-\frac{\beta}{2}\}$. 即 ρ 的分布近似于 $N(0,1)$,从而

$$P_r\{\theta_1<\theta<\theta_2\}=P_r\{t_1<t<t_2\}\approx\Phi(\rho_2)-\Phi(\rho_1),$$

其中 $\theta_i=(2\pi)^{-\frac{p}{2}}|\Lambda|^{-\frac{1}{2}}\int_{\underset{\sim}{z}<t_i\underset{\sim}{1}}\cdots\int\exp\{-\frac{1}{2}\underset{\sim}{z}^{\tau}\Lambda^{-1}\underset{\sim}{z}\}d\underset{\sim}{z},\rho_i=\frac{t_i-E(t)}{\sqrt{D(t)}},\ i=1,2.$

如令 $0<\alpha<1,\Phi(\rho_\alpha)=\alpha,t_\alpha=E(t)+\rho_\alpha\sqrt{D(t)}$,则 θ 的置信水平 $1-\alpha$ 的近似置信下界为

$$\theta_\alpha(2\pi)^{-\frac{p}{2}}|\Lambda|^{-\frac{1}{2}}\int_{\underset{\sim}{z}<t_\alpha\underset{\sim}{1}}\cdots\int\exp\{-\frac{1}{2}\underset{\sim}{z}^{\tau}\Lambda^{-1}\underset{\sim}{z}\}d\underset{\sim}{z},$$

即 $P_r\{\theta_\alpha<\theta\}=P_r\{t_\alpha<t\}=1-\alpha.$

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Bayes Estimation on the Reliability of Normal-Normal Structural Series System

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Abstract

We discuss the estimation problem on the reliability $\theta=P_r(x>Y)$ of Normal-Normal structural series system. The posterror mean of θ is considered here as its point estimator. In some caces, the Bayes confidence bound of θ is given.

Keywords reliability, Bayes confidence bound.