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BMO Boundedness of Generalized Littlewood-Paley Functions *

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Abstract: It is proved that the image of a BMO function under the generalized Littlewood-Paley functions is either equal to infinity almost everywhere or in BMO.

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For $x \in R^n$ and $t > 0$, the Poission kernel for the upper halfphane , R_+^{n+1}, is

$$P(x,t) = c_n \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}},$$

the Poission integral of $f(x) \in L^1_{\text{loc}}(R^n)$ is

$$f(x,t) = [f * P(\cdot,t)](x) = \int_{R^n} P(x-y,t) f(y) \mathrm{d}y$$

and the gradient of $f(x,y)$ is

$$\nabla f(x,t) = (\frac{\partial f}{\partial x_1}(x,t), \cdots, \frac{\partial f}{\partial x_n}(x,t), \frac{\partial f}{\partial t}(x,t)).$$

The Littlewood-Paley function $g(f)$ is defined by

$$g(f) = (\int_0^\infty t|\nabla f(x,t)|^2 \mathrm{d}t)^{\frac{1}{2}}.$$

Wang[1] prove that for $f \in \mathrm{BMO}(R^n)$, either $g(f)(x) = \infty$ almost everywhere or $g(f)(x) < \infty$ almost everywhere and there is a constant C depending only on n such that

$$\|g(f)\|_* \le C\|f\|_*,$$

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where $\|\cdot\|_*$ is the norm in BMO. In this note, we will generalize this result to a more general case.

Let $\psi(x)$ is function defined on R^n, for $\beta > 0$ and $\gamma > 0$, satisfying

(1) $|\psi(x)| \leq \frac{C}{(1+|x|)^{n+\gamma}}, x \in R^n$

(2) for any $x, y \in R^n, |x-y| \leq \frac{1}{2}(1+|x|)$,

$$|\psi(x) - \psi(y)| \leq C(\frac{|x-y|}{1+|x|})^\beta \frac{1}{(1+|x|)^{n+\gamma}},$$

(3) $\int_{R^n} \psi(x)\mathrm{d}x = 0$.

We define the generalized Littlewood-Paley function by

$$G(f)(x) = (\int_0^\infty |f * \psi_t(x)|^2 \frac{\mathrm{d}t}{t})^{\frac{1}{2}},$$

where $f(x) \in L^1_{\text{loc}}(R^n)$, $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$.

Lemma 1 *Let $f \in BMO(R^n)$, $\gamma > 0$ and $p \geq 1$, let Q be a cube centered at x and have edge length r. There is a constant C depending on n, γ and p so that for $t > 0$*

$$(\int_{R^n} \frac{|f(y) - f_Q|^p}{(|y-x|+t)^{n+\gamma}} \mathrm{d}y)^{\frac{1}{p}} \leq Ct^{-\frac{\gamma}{p}}(1 + |\ln[\frac{t}{r}]|)\|f\|_*.$$

The proof of this Lemma is similar to the Lemma 1.1 in [2]. The following is our main result.

Theorem 2 *Let $f \in BMO(R^n)$. Either $G(f)(x) = \infty$ almost everywhere or $G(f)(x) < \infty$ almost everywhere and there is a constant C depending only on n such that*

$$\|G(f)\|_* \leq C\|f\|_*.$$

Proof Suppose $G(f)(x) \neq \infty$ almost everywhere. The $E = \{x : G(f)(x) < \infty\}$ has positive measure. Let $\bar{x}$ be a point of density of E, and Q be any cube centered at $\bar{x}$ and set $f_Q = \frac{1}{|Q|}\int_Q f(t)\mathrm{d}t$. Write f as

$$f(x) = f_Q + [f(x) - f_Q]\chi_Q(x) + [f(x) - f_Q]\chi_{Q^c}(x) = f_Q + g_Q(x) + h_Q(x),$$

where Q^c denote the complement of Q. Since f_Q is a constant , $G(f_Q)$ is identically 0. Thus $G(f_Q)$ is in BMO with BMO norm equal to 0. Therefore,

$$G(f) \leq G(g_Q) + G(h_Q)$$

and

$$G(h_Q) \leq G(f) + G(g_Q).$$

Since $f \in \mathrm{BMO}(R^n)$, we have

$$\|g_Q\|_2 = (\int_Q |f(t) - f_Q|^2 \mathrm{d}t)^{\frac{1}{2}} \leq C|Q|^{\frac{1}{2}}\|f\|_*$$

and $g_Q \in L^2$. Thus, $G(g_Q)$ is finite almost everywhere. Therefore, $G(f)(x) < \infty$ at almost every point such that $G(h_Q)(x) < \infty$.

Let $d < 1$. Since $\bar{x}$ is a point of density of E and $G(g_Q)$ is finite almost everywhere, there is a point x' in dQ such that $G(f)(x'), G(g_Q)(x')$ and $G(h_Q)(x')$ are finite.

In the following Lemma 3 we will prove that for a sufficiently small d, there is a constant C so that for all $x \in dQ$,

(i) $G(h_Q)(x') < \infty \Rightarrow G(h_Q)(x) < \infty$,

(ii) $|G(h_Q)(x) - G(h_Q(x')| \le C\|f\|_*$.

Now we assume that (i) and (ii) are true. Fix a cube Q centered at $\bar{x}$. As above, there is an $x' \in dQ$ so that $G(h_Q)(x') < \infty$. By (i), $G(h_Q)(x)$ and $G(f)(x)$ is finite almost everywhere in dQ. Considering only cubes centered at $\bar{x}$ with edge length equal to a positive integer shows $G(f)$ is finite almost everywhere.

Now we prove that $\|G(f)\|_* \le C\|f\|_*$. Let Q' be any cube and set $Q = \frac{1}{d}Q'$. Choose a point $x' \in dQ$ so that $G(h_Q)(x')$ is finite. Then by (i) and (ii),

$$\begin{aligned}
&\frac{1}{|Q'|}\int_{Q'}|G(f)(x) - G(h_Q)(x')|\mathrm{d}x \\
&= \frac{1}{|Q'|}\int_{Q'}|G(g_Q + h_Q)(x) - G(h_Q)(x) + G(h_Q)(x) - G(h_Q)(x')|\mathrm{d}x \\
&\le \frac{1}{|Q'|}\int_{Q'}|G(g_Q)(x)|\mathrm{d}x + \frac{1}{|Q'|}\int_{Q'}|G(h_Q)(x) - G(h_Q)(x')|\mathrm{d}x \le C\|f\|_*.
\end{aligned}$$

So $\|G(f)\|_* \le C\|f\|_*$, the proof is complete.

Lemma 3 *Suppose $f \in BMO(R^n)$. Let Q be a cube with center $\bar{x}$ and edge length r. Set $d = \frac{1}{8\sqrt{n}}$. Suppose there is an $x' \in dQ$ so that $G(h_Q)(x') < \infty$. Then there is a constant C, depending only on n, such that*

$$G(h_Q)(x) < \infty$$

and

$$|G(h_Q)(x) - G(h_Q)(x')| \le C\|f\|_* \quad \text{for all} \quad x \in dQ.$$

The proof of lemma 3 is simple, the readers can refer to [2].

At last, we define the generalized area integral by

$$S(f)(x) = (\int\int_{\Gamma(x)}|f * \psi_t(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}})^{\frac{1}{2}},$$

where $\Gamma(x) = \{(y,t) \in (R^n,(0,\infty)) : |x - y| < t, \ \ t > 0\}$ and for $\lambda > 0$, we define other Littlewood-Paley g-function as

$$g_\lambda(f)(x) = \int_0^\infty\int_{R^n}(\frac{t}{t + |y - x|})^{n\lambda}|\psi_t * f(y)|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}})^{\frac{1}{2}}.$$

Then we can use the similar method to get the same results for $S(f)$ and $g_\lambda(f)$, the details are omited. The readers can refer to [2].

References:

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广义 Littlewood-Paley 函数的 BMO 有界性

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摘 要: 本文证明了 BMO 函数在广义 Littlewood-Paley 函数下的象或者几乎处处等于无穷或者属于 BMO.