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The Algebraic Properties of a Type of Infinite Lower Triangular Matrices Related to Derivatives *

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Abstract: In this paper, Jabotinsky matrices in [4, 5] are modified and a type of infinite lower triangular matrices $T(f)$ is discussed. Some algebraic properties of $T(f)$ are obtained and proved. Additionally, some inverse pairs and combinatorial identities associated with derivatives are obtained.

Key words: Jabotinsky matrix; combinatorial identity; inverse pair; derivative.

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0. Introduction

In [4, 5], Jabotinsky introduced a special kind of infinite lower matrices. In [3], J.L. Lavoie and R. Tremblay studied the inversions of formal power series and the related results in terms of the Jabotinsky matrices. In this paper, we combine the work of [1-7] to modify the Jabotinsky matrices and obtain some new identities and inverse relations related to derivatives.

1. The modified Jabotinsky matrix

Let $F(t)=\sum_{i=0}^{\infty}q_it^i\in\mathcal{L}(\mathcal{F})$ with $q_0=q\neq 0$, be a given formal power series(fps). The set $\mathcal{L}(\mathcal{F})$ is the totality of fps with coefficients $q_i\in\mathcal{F}$, a field of characteristic zero. Let $f(t)=tF(t)$. Then an infinite lower triangular matrix L called the modified Jabotinsky matrix of $f(t)$ is defined as follows:

$$L=\begin{pmatrix} L_{00} & & \\ L_{10} & L_{11} & \\ L_{20} & L_{21} & L_{22} \\ & \cdots & \end{pmatrix}=(L_{ij}),\ i\geq j,\ i,j=0,1,\cdots,$$

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where L_{ij} are generated by $f^j(t) = \sum_{i=j}^{\infty} L_{ij}t^i$, $j = 0, 1, 2, \cdots$, and from Taylor's theorem, $L_{ij} = \frac{1}{(i-j)!} D^{(i-j)} F^j(t) \mid_{t=0}$, $i \geq j$, with $D = \frac{d}{dt}$. Clearly, if $F(t) = 1$ then $L = I$, where I is a unit matrix of infinite order.

In this paper, we introduce a new notation $T = [T_{ij}]$ to represent the infinite lower triangular matrix $T = (T_{ij})$, $i \geq j$, $i, j = 0, 1, 2, \cdots$. So the modified Jabotinsky matrix of $f(t)$ can be denoted as $L = [L_{ij}]$.

2. Inverse pairs related to the modified Jabotinsky matrix

Let $g(t) = tG(t)$ and let $M = [M_{ij}]$ be its modified Jabotinsky matrix. From (3.1) in [3], we have

Theorem 2.1 *Let $A = (a_0, a_1, a_2, \cdots)^T$ and $B = (b_0, b_1, b_2, \cdots)^T$. If $f(g(t)) = t = g(f(t))$, then we have $ML = I = LM$ and the corresponding inverse pair of matrix relations*

$$\begin{cases} A = LB, \\ B = MA. \end{cases}$$

From Theorem 2.1, we obtain a number of inverse pairs of matrix relations in Table 1, where L_{ij}, M_{ij} can be obtained by the method in [3].

Table 1:

$F(t)$	$G(t)$	L_{ij}	M_{ij}
$1 + xt$	$\frac{2}{1+(1+4xt)^{\frac{1}{2}}}$	$x^{i-j}\binom{j}{i-j}$	$(-x)^{i-j}\frac{j}{i}\binom{2i-j-1}{i-1}$
$(1 + xt)^{-1}$	$(1 - xt)^{-1}$	$(-x)^{i-j}\binom{i-1}{j-1}$	$x^{i-j}\binom{i-1}{j-1}$
$\frac{\ln(1+t)}{t}$	$\frac{e^t-1}{t}$	$\frac{j!}{i!}S_1(i,j)$	$\frac{j!}{i!}S_2(i,j)$

The $S_1(i,j)$ and $S_2(i,j)$ are the Stirling numbers of both kinds (see[2, 3]).

Theorem 2.2 *Let $\begin{cases} A = M_1B, \\ B = N_1A, \end{cases}$ and $\begin{cases} A = M_2B, \\ B = N_2A, \end{cases}$ be two inverse pairs. Then we have the following inverse pair*

$$\begin{cases} A = N_2M_1B, \\ B = N_1M_2A. \end{cases} \qquad (*)$$

Proof. Since $\begin{cases} A = M_1B, \\ B = N_1A, \end{cases}$ and $\begin{cases} A = M_2B, \\ B = N_2A, \end{cases}$ are two inverse pairs, we have $M_1N_1 = I$, $M_2N_2 = I$. Hence $M_1N_1M_2N_2 = I$, and $(N_2M_1)^{-1} = N_1M_2$.
This completes the proof. □

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Example 2.1 Consider the inverse pairs in Table 1

$$\begin{cases} A = [x^{i-j}\binom{j}{i-j}]B, \\ B = [(-x)^{i-j}\frac{j}{i}\binom{2i-j-1}{i-1}]A, \end{cases}$$

and

$$\begin{cases} A = [\frac{j!}{i!}S_1(i,j)]B, \\ B = [\frac{j!}{i!}S_2(i,j)]A. \end{cases}$$

From(*), we have the following inverse pair

$$\begin{cases} A = [\frac{j!}{i!}S_2(i,j)][x^{i-j}\binom{j}{i-j}]B, \\ B = [(-x)^{i-j}\frac{j}{i}\binom{2i-j-1}{i-1}][\frac{j!}{i!}S_1(i,j)]A. \quad \square \end{cases}$$

Clearly, we can obtain many new inverse pairs using (*).

3. Some algebraic properties of $T(f)$

From now on, let $f(t)$, $g(t)$ be two derivable functions of infinite order. We substitute $T_{ij} = \frac{D^{(i-j)}f(t)}{(i-j)!}$ for $L_{ij} = \frac{D^{(i-j)}F^j(t)}{(i-j)!}$ in the modified Jabotinsky matrices and obtain a kind of infinite lower triangular matrices, denoted by $T(f)$, as follows:

$$T(f) = \begin{pmatrix} f(t) & & & & \\ Df(t) & f(t) & & & \\ \frac{D^{(2)}f(t)}{2!} & Df(t) & f(t) & & \\ & \cdots & & & \\ \frac{D^{(n-1)}f(t)}{(n-1)!} & \frac{D^{(n-2)}f(t)}{(n-2)!} & \frac{D^{(n-3)}f(t)}{(n-3)!} & \cdots & f(t) \\ & \cdots & & \cdots & \end{pmatrix} = [T_{ij}],$$

where $T_{ij} = \frac{D^{(i-j)}f(t)}{(i-j)!}$, $i \geq j$, $i,j = 0,1,2,\cdots$. If $T_{ij} = \frac{D^{(i-j)}f(t)}{(i-j)!} \mid_{t=t_0}$, $i \geq j$, $i,j = 0,1,2,\cdots$ in $T(f)$, we write $T(f)\mid_{t=t_0}$ for the corresponding matrix. Moreover, let $T_n(f)$ denote the matrix of order n with the first n rows and n columns in $T(f)$.

Let

$$P = \begin{pmatrix} 0 & & & & \\ p_1 & 0 & & & \\ & p_2 & 0 & & \\ & & p_3 & 0 & \\ & & \cdots & \cdots & \end{pmatrix},$$

and $\langle p_j\rangle_m = p_jp_{j+1}p_{j+2}\cdots p_{j+m-1}$. Then $P^m = \text{subdiag}_{i-j=k}[\langle p_1\rangle_m, \langle p_2\rangle_m, \cdots]$, where $\text{subdiag}_{i-j=m}[\langle p_1\rangle_m, \langle p_2\rangle_m, \cdots]$ says that all elements in P^m are zero except for the elements $\langle p_1\rangle_m, \langle p_2\rangle_m, \cdots$, in m^{th} subdiag. Let $p_1 = p_2 = \cdots = 1$. We obtain a series of matrices $Q^m = \text{subdiag}_{i-j=m}[1,1,\cdots]$.

From [7], we have Theorem 3.1.

Theorem 3.1 *For any integer $i > 0$, we have*

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(1) $T_n(f)T_n(g) = T_n(fg)$; (2) $T_n^i(f) = T_n(f^i)$.

Theorem 3.2 $T(f) = \sum_{k=0}^{\infty} \frac{Q^k}{k!} D^{(k)} f(t)$.

Proof $T(f) = \sum_{k=0}^{\infty} \frac{1}{k!} D^{(k)} f(t) \mathrm{subdiag}_{i-j=k}[1, 1, \cdots] = \sum_{k=0}^{\infty} \frac{Q^k}{k!} D^{(k)} f(t)$. □

Theorem 3.3

(1) $T(f+g) = T(f) + T(g)$; (2) $T(fg) = T(f)T(g) = T(g)T(f)$.

Proof Here we only prove (2). From Theorem 3.2, we have

$$\begin{aligned} T(fg) &= \sum_{k=0}^{\infty} \frac{Q^k}{k!} D^{(k)}(f(t)g(t)) = \sum_{k=0}^{\infty} \frac{Q^k}{k!} \sum_{l=0}^{k} \binom{k}{l} D^{(l)} f(t) D^{(k-l)} g(t) \\ &= \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} Q^k \frac{D^{(l)} f(t)}{l!} \frac{D^{(k-l)} g(t)}{(k-l)!} = \sum_{l=0}^{\infty} \frac{D^{(l)} f(t)}{l!} \sum_{k=l}^{\infty} Q^k \frac{D^{(k-l)} g(t)}{(k-l)!} \\ &= \sum_{l=0}^{\infty} \frac{Q^l}{l!} D^{(l)} f(t) \sum_{k=0}^{\infty} \frac{Q^k}{k!} D^{(k)} g(t) = T(f)T(g) \\ &= T(g)T(f). \quad \square \end{aligned}$$

Corollary 3.4

(1) $T(af) = aT(f)$, *where a is a real number,*

(2) For any integer $i > 0$, $T^i(f) = T(f^i)$,

(3) $T(f^2 - g^2) = T(f^2) - T(g^2) = T^2(f) - T^2(g) = (T(f) - T(g))(T(f) + T(g))$
$= T(f-g)T(f+g)$,

(4) $T(f^2 + g^2) = T(f^2) + T(g^2) = T^2(f) + T^2(g)$,

(5) $T(\sum_{i=0}^{k} \binom{k}{i} f^i g^{k-i}) = T((f+g)^k) = T^k(f+g) = (T(f) + T(g))^k$
$= \sum_{i=0}^{k} \binom{k}{i} T^i(f) T^{k-i}(g) = \sum_{i=0}^{k} \binom{k}{i} T(f^i) T(g^{k-i})$,

(6) $T((f-g)^k) = T(\sum_{i=0}^{k} (-1)^{k-i} f^i g^{k-i}) = T^k(f-g) = (T(f) - T(g))^k$
$= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} T^i(f) T^{k-i}(g) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} T(f^i) T(g^{k-i})$,

(7) $T(w(f(t))) = w(T(f))$, *where $w(t)$ is a polynomial,*

(8) $T(1 + \frac{g}{f}) = I + T(\frac{g}{f})$.

4. Inverse pairs related to $T(f)$

For $T(1) = I$, from Theorem 3.3, we have

Theorem 4.1 *If $f(t)g(t) = 1$, then we have inverse pair*

$$\begin{cases} A = T(f)B, \\ B = T(g)A. \end{cases}$$

Example 4.1 Let $f(t) = (1+t)^a$, where $a \neq 0$ is a real number, and $g(t) = (1+t)^{-a}$. Then we have the inverse pairs

$$\begin{cases} A = T((1+t)^a)B, \\ B = T((1+t)^{-a})A, \end{cases}$$

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and

$$\begin{cases} A = T((1+t)^a)\mid_{t=0} B = [\binom{a}{i-j}]B, \\ B = T((1+t)^{-a})\mid_{t=0} A = [(-1)^{i-j}\binom{a+i-j-1}{i-j}]A. \end{cases} \quad \square$$

In Table 2, we show a number of other inverse pairs.

Table 2:

1.	$\begin{cases} A = T(e^{at})B, \\ B = T(e^{-at})A \end{cases}$,	and	$\begin{cases} A = [\frac{a^{i-j}}{(i-j)!}]B, \\ B = [\frac{(-a)^{i-j}}{(i-j)!}]A. \end{cases}$
2.	$\begin{cases} A = T((1-t)^a)B, \\ B = T((1-t)^{-a})A, \end{cases}$	and	$\begin{cases} A = [(-1)^{i-j}\binom{a}{i-j}]B, \\ B = [\binom{i-j+a-1}{i-j}]A. \end{cases}$
3.	$\begin{cases} A = T(\frac{\tan t}{t})B, \\ B = T(t\cot t)A, \end{cases}$	and	$\begin{cases} A = [T_{ij}(\frac{\tan t}{t})\mid_{t=0}]B, \\ B = [T_{ij}(t\cot t)\mid_{t=0}]A. \end{cases}$

In Table 2, a is a real number, and

$$T_{ij}(\frac{\tan t}{t})\mid_{t=0} = \begin{cases} \frac{(-1)^{\frac{i-j}{2}}2^{i-j+2}(2^{i-j+2}-1)B_{i-j+2}}{(i-j+2)!} & i-j \text{ even}, \\ 0 & i-j \text{ odd}, \end{cases}$$

and

$$T_{ij}(t\cot t)\mid_{t=0} = \begin{cases} \frac{(-4)^{i-j}B_{i-j}}{(i-j)!} & i-j \text{ even}, \\ 0 & i-j \text{ odd}, \end{cases}$$

where B_n are Bernoulli numbers defined by $\frac{t}{e^t-1} = \sum_{n\geq 0}\frac{B_n t^n}{n!}$.

5. Combinatorial identities related to $T_n(f)$

Let $e_k(0 \leq k \leq n)$ be the unit vector in $\mathbf{R}^{n\times 1}$ and aslo let

$$e_k(g) = k!(g(t), Dg(t), \frac{D^{(2)}g(t)}{2!}, \cdots, \frac{D^{(n-1)}g(t)}{(n-1)!})^T.$$

Then we have

Lemma 5.1 *For any integers $i > 0$ and $0 \leq k \leq n-1$, we have*

$$e_{k+1}^T T_n^i(f)e_k(g) = D^{(k)}(f^i(t)g(t)).$$

Proof

$$e_{k+1}^T T_n^i(f)e_k(g) = e_{k+1}^T T_n(f^i)e_k(g) = \sum_{j=0}^{k}\frac{D^{(j)}f^j(t)}{j!}k!\frac{D^{(k-j)}g(t)}{(k-j)!} = D^{(k)}(f^i(t)g(t)). \quad \square$$

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Theorem 5.2 *Let $I_n(f) = \text{diag}(f(t), \cdots, f(t))$. If k, l are two positive integers and $k \le l-1$, then we have $\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}f^{l-i}(t)D^{(k)}(f^i(t)g(t)) = 0$.*

Proof From Lemma 5.1, we have

$$\sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}f^{l-i}(t)D^{(k)}(f^i(t)g(t)) = \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}f^{l-i}(t)\mathrm{e}_{k+1}^T T_n^i(f)\mathrm{e}_k(g)$$

$$= \mathrm{e}_{k+1}^T \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}f^{l-i}(t)T_n^i(f)\mathrm{e}_k(g) = \mathrm{e}_{k+1}^T \sum_{i=0}^{l}(-1)^{l-i}\binom{l}{i}I_n^{l-i}(f)T_n^i(f)\mathrm{e}_k(g)$$

$$= \mathrm{e}_{k+1}^T(T_n(f) - I_n(f))^l \mathrm{e}_k(g) = 0. \quad \square$$

Lemma 5.3 *For any integer $n > 1$, we have $(T_n(f) - I_n(f))^{n-1} = M_n(f)$, where $M_n(f)$ is a matrix of order n, in which all elements are equal to zero except for $(M_n(f))_{n,1} = (D(f(t)))^{n-1}$.*

Theorem 5.4 *For any integer $n > 1$, we have*

$$\sum_{i=0}^{n-1}(-1)^{n-1-i}\binom{n-1}{i}f^{n-1-i}(t)D^{(n-1)}(f^i(t)g(t)) = (n-1)!(D(f(t)))^{n-1}g(t).$$

The proofs of Lemma 5.3 and Theorem 5.4 are similar to the proofs of Lemma 4.3 and Theorem 4.4 in [2].

References:

[1] BAYAT M, TEIMOORI H. *The linear algebra of the generalized Pascal functional matrix* [J]. Linear Algebra and its Applications, 1999, **295**: 81–89.

[2] ZHAO X, WANG T. *The algebraic properties of the generalized Pascal functional matrices associated with the exponential families* [J]. Linear Algebra and its Applications, 2000, **318**: 45–52.

[3] LAVOIE J L, TREMBLAY R. *The Jabotinsky matrix of a power series* [J]. SIAM J. Math. Anal., 1981, **12**(6): 819–824.

[4] JABOTINSKY E. *Representation of functions by matrices. Application to Faber polynomials* [J]. Proc. Amer. Math. Soc., 1953, **4**: 546–553.

[5] JABOTINSKY E. *Analytic iteration* [J]. Trans. Amer. Math. Soc., 1963, **108**: 457–477.

[6] VEIN P R. *Identities among certain triangular matrices* [J]. Linear Algebra and its Applications, 1986, **82**: 27–79.

[7] KAIMAN D, UNGAR A. *Combinatorial and functional identities in one-parameter matrices* [J]. The American Mathematical Monthly, 1987, **94**(1): 21–35.

一类无穷下三角矩阵的代数性质

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摘 要: 修正了 [4,5] 中的 Jabotinsky 矩阵, 得到并证明了一类无穷下三角矩阵 $T(f)$ 的一些性质, 最后, 导出了一些与导数相关的反演关系和组合恒等式.

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