Journal of Mathematical Research & Exposition
Vol.22, No.4, 567–574, November, 2002

Oscillation of Neutral Differential Equations with "Integrally Small" Coefficients *

ZHANG Xiao-sheng[1], *WANG Duo*[2]
(1. Dept. of Math., Captial Normal University, Beijing 100037, China;
2. Dept. of Math., Peking University, Beijing 100871, China)

Abstract: Sufficient conditions for the oscillation of the neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-R(t)x(t-r)]+P(t)x(t-\tau)-Q(t)x(t-\delta)=0,$$

where $P,Q,R\in C([t_0,\infty),R^+)$, and $r,\ \tau,\ \delta\in(0,\ \infty)$, are obtained for the case where former results can not be applied.

Key words: differential equation; delay; neutral; oscillation.

Classification: AMS(2000) 34K15/CLC O175.2

Document code: A **Article ID:** 1000-341X(2002)04-0567-08

1. Introduction

Consider the following neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-R(t)x(t-r)]+P(t)x(t-\tau)-Q(t)x(t-\delta)=0, \tag{1}$$

and, the corresponding differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-R(t)x(t-r)]+P(t)x(t-\tau)-Q(t)x(t-\delta)\le 0, \tag{1'}$$

where

$$\begin{aligned}&P,Q,R\in C([t_0,\infty),R^+),\ \text{and}\ r,\ \tau,\ \delta\in(0,\ \infty),\ \ \tau\ge\delta,\\ &\overline{P}(t)\equiv P(t)-Q(t-\tau+\delta)\ge 0\ \text{and not identically zero.}\end{aligned} \tag{2}$$

Recently, the oscillation of equation (1) has been studied in many papers(see [1-6]), and a lot of interesting results have been given in the literature. It is valuable to cite the

*Received date: 2000-09-13
Foundation item: Supported by NNSFC, RFDP and Yunnan Edu. Project
Biography: ZHANG Xiao-sheng (1963-), male, born in Hequ county, Shanxi province, Ph.D., Associate Professor.

following results in [6].

Theorem A[6] *Assume that (2) and*

$$\int_{t_0}^{\infty} s\overline{P}(s)\int_{s}^{\infty}\overline{P}(u)\mathrm{d}u\mathrm{d}s=\infty \tag{*}$$

hold and that

$$R(t)+\int_{t-\tau+\delta}^{t}Q(u)\mathrm{d}u\equiv 1. \tag{**}$$

Then every solution of equation (1) oscillates.

Theorem B[6] *Assume that (2) ,(∗) and*

$$R(t)+\int_{t-\tau+\delta}^{t}Q(u)\mathrm{d}u\geq 1$$

hold and that

$$R(t-\tau)\overline{P}(t)\leq h_1\overline{P}(t-r). \tag{†}$$

Also suppose that $Q(t)/\overline{P}(t)$ is nondecreasing and satisfies

$$\overline{P}(t)Q(t-\tau)\leq h_2\overline{P}(t-\delta), \tag{††}$$

where h_1, h_2 are nonegative constants and satisfy

$$h_1+h_2(\tau-\delta)=1.$$

Then every solution of equation (1) oscillates.

The main purpose of this paper is to investigate the oscillation of (1) and (1)′ when the assumptions (∗∗), (†) and (††) can't be satisfied. Some examples are given to illustrate the application of our results. Throughout this paper we always assume that (1) and (1)′ satisfy the condition (2).

2. Some lemmas

The following Lemma 1 is a functional integral inequality, and is important in the proof of our main results.

Lemma 1 *Assume that $A\geq 0$, $r>0$, $a<b<\infty$ and that $f\in C([a,b],R^+)$, $g\in C[R^+,R^+]$. If $y\in C([a-r,b],R^+)$ satisfies*

$$y(s)\leq g[y(s-r)]+A+\int_a^s f(u)y(u)\mathrm{d}u,\ a\leq s\leq b, \tag{3}$$

then

$$y(s)\leq g[y(s-r)]+e^{F(s)}[A+\int_a^s f(u)g[y(u-r)]\mathrm{d}u],\ a\leq s\leq b, \tag{4}$$

where $F(s)=\int_a^s f(u)\mathrm{d}u$.

Proof First we claim that (4) holds for $s \in [a, a+r]$. In fact, since $C^1[a, a+r]$ is dense in $C[a, a+r]$, we can choose, for any $\epsilon > 0$, $g_1 \in C^1([a, a+r], R^+)$ such that $|g[y(s-r)] - g_1(s)| < \epsilon$, $a \le s \le a+r$. From (3) it follows that

$$y(s) \le g_1(s) + A + \epsilon + \int_a^s f(u)y(u)\mathrm{d}u, \;\; a \le s \le a+r.$$

By Gronwall inequality

$$\begin{aligned} y(s) \le & e^{F(s)}[g_1(a) + A + \epsilon + \int_a^s g_1'(u)e^{-F(u)\mathrm{d}u}] \\ \le & g_1(s) + e^{F(s)}[A + \epsilon + \int_a^s f(u)g_1(u)\mathrm{d}u] \\ \le & g[y(s-r)] + e^{F(s)}[A + \int_a^s f(u)g[y(u-r)]\mathrm{d}u] + \\ & \epsilon + \epsilon e^{F(a+r)}[1 + \int_a^{a+r} f(u)\mathrm{d}u], \; a \le s \le a+r, \end{aligned}$$

which, by the fact that $\epsilon > 0$ can be arbitrarily small, implies that (4) holds for $s \in [a, a+r]$.

For $a+r \le s \le a+2r$, rewrite (3) into the following inequality,

$$y(s) \le g[y(s-r)] + A + \int_a^{a+r} f(u)y(u)\mathrm{d}u + \int_{a+r}^s f(u)y(u)\mathrm{d}u, \; a+r \le s \le a+2r.$$

By applying the conclusion obtained above to this inequality on $[a+r, a+2r]$, w e find

$$y(s) \le g[y(s-r)] + e^{F_1(s)}[A + \int_a^{a+r} f(u)y(u)\mathrm{d}u + \int_{a+r}^s f(u)g[y(u-r)]\mathrm{d}u], \tag{5}$$

where $F_1(s) = \int_{a+r}^s f(u)\mathrm{d}u$. Noticing that (4) holds for $a \le s \le a+r$, we obtain

$$\int_a^{a+r} f(u)y(u)\mathrm{d}u \le (e^{F(a+r)} - 1)A + e^{F(a+r)}\int_a^{a+r} f(u)g[y(u-r)]\mathrm{d}u,$$

which, combining with (5), implies that

$$y(s) \le g[y(s-r)] + e^{F(s)}[A + \int_a^s f(u)g[y(u-r)]\mathrm{d}u], \; a+r \le s \le a+2r.$$

That is (4) holds for $s \in [a+r, a+2r]$. Set $N_1 = [\frac{b-a}{r}] + 1$. Repeating above process N_1 times, we can get that (4) holds on $[a, b]$. The proof is complete.

Lemma 2 *Assume that for sufficiently large* t,

$$0 \le R(t) \le 1 \tag{6}$$

and

$$P(t) \ge Q(t - \tau + \delta)\exp[\frac{t}{\tau}\int_{t-\tau+\delta}^t Q(u)\mathrm{d}u] \tag{7}$$

hold and that

$$t\int_{t-\tau+\delta}^{t} Q(u)\mathrm{d}u \text{ is bounded on } [t_0,\ \infty). \tag{8}$$

Let $x(t)$ be an eventually positive solution of inequality $(1)'$ *and*

$$z(t) = x(t) - R(t)x(t-r) - \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)\mathrm{d}s. \tag{9}$$

Then eventually

$$z'(t) \le 0, \quad z(t) > 0. \tag{10}$$

Proof From $(1)'$ and (2), it follows immediately that

$$z'(t) \le -\overline{P}(t)x(t-\tau) \le 0. \tag{11}$$

suppose that the fact $z(t) > 0$ eventually does not hold. Then from (11), there exists a positive number $\alpha > 0$ such that eventually

$$x(t) > 0 \ \text{ and } \ z(t) \le -\alpha. \tag{12}$$

Choose $\bar{t} \ge \max\{t_0 + \overline{\tau}, \delta + r\}$, where $\overline{\tau} = r + \tau$, such that (2), (6), (7), (11) and (12) hold for $t \ge \bar{t} - \overline{\tau}$. Let $M = \max\{x(s),\ \bar{t} - r - \tau + \delta \le s \le \bar{t}\}$. For any $t \ge \bar{t} + \delta$, define $n(t) = [\frac{t-\bar{t}-\delta}{\tau}] + 1$. Integrating (11) on $[t-\tau, t]$, and noticing (6), (9) and (12), we get for $t \ge \bar{t} + \delta$,

$$x(t) \le -\alpha + x(t-r) - \int_{t-\tau}^{t} \overline{P}(u)x(u-\tau)\mathrm{d}u + \int_{t-\tau+\delta}^{t} Q(u)x(u-\delta)\mathrm{d}u. \tag{13}$$

For $t - \tau + \delta \le s \le t$, define

$$\begin{aligned} \overline{q}(s) =& \int_{t-\tau+\delta}^{s} Q(u)\mathrm{d}u, \\ x_k(s) =& e^{-\overline{q}(s)}x_{k-1}(s-r), \ \ x_0(s) = x(s-\delta),\ k = 1, 2, \cdots, \\ A(t) =& \int_{t-\tau}^{t} Q(u-\tau+\delta)x(u-\tau)\mathrm{d}u. \end{aligned}$$

By (13) and $t - \tau + \delta \le s \le t$, we have

$$\begin{aligned} x_0(s) \le& x_0(s-r) + \int_{t-2\tau+\delta}^{s} Q(u)x_0(u)\mathrm{d}u \\ \le& x_0(s-r) + A(t) + \int_{t-\tau+\delta^{s}} Q(u)x_0(u)\mathrm{d}u, \end{aligned} \tag{14}$$

which, by Lemma 1, yields that for $t - \tau + \delta \le s \le t$,

$$\begin{aligned} x_0(s) \le& x_0(s-r) + e^{\overline{q}(s)}[A(t) + \int_{t-\tau+\delta}^{s} e^{-\overline{q}(u)}Q(u)x_0(u-r)\mathrm{d}u] \\ \le& x_0(s-r) + e^{\overline{q}(s)}[A(t) + \int_{t-\tau+\delta}^{s} Q(u)x_1(u)\mathrm{d}u]. \end{aligned} \tag{15}$$

Noticing that $x(u-\delta)=x_0(u)$, $t-\tau+\delta\le u\le t$, substituting (15) into (13) and changing the integral order, one can get for $t\ge\bar{t}+\delta$

$$x(t)\le-\alpha+x(t-r)-\int_{t-\tau}^{t}P(u)x(u-\tau)\mathrm{d}u+ \\ A(t)\exp[\overline{q}(t)]+\exp[\overline{q}(t)]\int_{t-\tau+\delta}^{t}Q(u)x_1(u)\mathrm{d}u. \quad (13.1)$$

From (15) we have

$$x_0(s-r)\le x_0(s-2r)+e^{\overline{q}(s)}[A(t)+\int_{t-\tau+\delta}^{s}Q(u)x_1(u)\mathrm{d}u],$$

which, devided by $e^{\overline{q}(s)}$, implies that

$$x_1(s)\le e^{-\overline{q}(s)}x_0(s-2r)+A(t)+\int_{t-\tau+\delta}^{s}Q(u)x_1(u)\mathrm{d}u \\ \le x_1(s-r)+A(t)+\int_{t-\tau+\delta}^{s}Q(u)x_1(u)\mathrm{d}u. \quad (14.1)$$

Again by using Lemma 1 in (14.1), we get

$$x_1(s)\le x_1(s-r)+e^{\overline{q}(s)}[A(t)+\int_{t-\tau+\delta}^{s}Q(u)x_2(u)\mathrm{d}u], \quad (15.1)$$

Substituting (15.1) into (13.1) we get for $t\ge\bar{t}+\delta$

$$x(t)\le-\alpha+x(t-r)-\int_{t-\tau}^{t}P(u)x(u-\tau)\mathrm{d}u+ \\ \exp[2\overline{q}(t)]A(t)+\exp[2\overline{q}(t)]\int_{t-\tau+\delta}^{t}Q(u)x_2(u)\mathrm{d}u. \quad (13.2)$$

Repeating above process $n(t)$ times we get for $t\ge\bar{t}+\delta$

$$x(t)\le-\alpha+x(t-r)-\int_{t-\tau}^{t}P(u)x(u-\tau)\mathrm{d}u+ \\ e^{n(t)\overline{q}(t)}A(t)+e^{n(t)\overline{q}(t)}\int_{t-\tau+\delta}^{t}Q(u)x_{n(t)}(u)\mathrm{d}u. \quad (13.n(t))$$

Notice that for $t\ge\bar{t}+\delta>r$ and $u\in[t-\tau+\delta,t]$, $\bar{t}-r-\tau+\delta\le u-n(t)r-\delta\le\bar{t}$ and $x_{n(t)}(u)\le x(u-n(t)r-\delta)\le M$. In addition $n(t)\le\frac{t-\bar{t}-\delta}{r}+1\le\frac{t}{r}$. We have from (13.n(t))

$$x(t)\le-\alpha+x(t-r)-\int_{t-\tau}^{t}[P(u)-e^{\frac{t}{r}\overline{q}(t)}Q(u-\tau+\delta)]x(u-\tau)\mathrm{d}u+ \\ Me^{\frac{t}{r}\overline{q}(t)}\int_{t-\tau+\delta}^{t}Q(u)\mathrm{d}u.$$

Since (7) and (8) implies that $e^{\frac{t}{r}\overline{q}(t)}\int_{t-\tau+\delta}^{t} Q(u)\mathrm{d}u \to 0$ $(t \to \infty)$, we get from above inequality and (7), for sufficiently large t

$$x(t) \le -\alpha/2 + x(t-r).$$

The rest of the proof is similar to that of Lemma 1 in [6] and is omitted. The proof is complete.

Lemma 3[6] *Assume that*

$$R(t) + \int_{t-\tau+\delta}^{t} Q(s)\mathrm{d}s \ge 1 \tag{16}$$

and

$$\int_{t_0}^{\infty} s\overline{P}(s) \int_{s}^{\infty} \overline{P}(u)\mathrm{d}u\mathrm{d}s = \infty. \tag{17}$$

Let $x(t)$ be an eventually positive solution of inequality $(1)'$ and $z(t)$ be defined by (9). Then eventually

$$z'(t) \le 0, \quad z(t) < 0.$$

3. Main results

In this section we will apply the lemmas in Section 2 to establish several criteria for the oscillation of all solutions of equation (1) and inequality $(1)'$. The following Theorem 1 is an immediate consequence of Lemma 2 and Lemma 3.

Theorem 1 *Assume that (6)–(8), (16) and (17) hold. Then inequality $(1)'$ has no positive solution.*

Corollary 1 *Assume that (6), (8), (16) and (17) hold and that*

$$Q(t-\tau+\delta) = \mathrm{o}(P(t)), (t \to \infty). \tag{18}$$

Then inequality $(1)'$ has no positive solution.

Proof It is clear that (18) implies (7). By Theorem 1, inequality $(1)'$ has no positive solution. The proof is complete.

Example 1 Consider the neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - (1-t^{-3/2})x(t-r)] + t^{-\alpha}x(t-r) - t^{-\alpha'}x(t-\delta) = 0, \tag{19}$$

where $1 < \alpha < \alpha' < 3/2$, $\delta > 0$, $r = \delta + 1$. It is clear that equation (19) can't satisfy the condition (**) of Theorem A[6]. On the other hand, the condition (6), (8) and (16)–(18) hold for (19). By Corollary 1, every solution of equation (19) oscillates.

The following Theorem 2 is a comparison theorem.

Theorem 2 *Assume that (16) and (17) hold and that*

$$R(t-\tau)\overline{P}(t) \le \overline{P}(t-r). \tag{20}$$

Further suppose that $Q(t)/\overline{P}(t+\tau-\delta)$ is nonincreasing and that there exists $h>0$ such that

$$\overline{P}(t) \le h\overline{P}(t-\tau). \tag{21}$$

If the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)-y(t-r)]+[\overline{P}(t)+hQ(t-2\tau+\delta)]y(t-\tau)-hQ(t-2\tau+\delta)y(t-\delta)\le 0 \tag{22}$$

has no positive solution, then every solution of eqution (1) oscillates.

Proof Suppose, otherwise , that equation (1) has an eventually positive solution $x(t)$. Let $z(t)$ be defined by (9). From Lemma 3 $z(t)<0$. By (20) and (21), for sufficiently large t, we have

$$\begin{aligned}
z'(t) &= -\overline{P}(t)x(t-\tau)\\
&= -\overline{P}(t)[z(t-\tau)+R(t-\tau)x(t-r-\tau)+\int_{t-\tau+\delta}^{t}Q(s-\tau)x(s-\delta-\tau)\mathrm{d}s]\\
&\ge -\overline{P}(t)z(t-\tau)-\overline{P}(t-r)x(t-r-\tau)-\\
&\quad \overline{P}(t)\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)}{\overline{P}(s-\delta)}[-z'(s-\delta)]\mathrm{d}s\\
&\ge -\overline{P}(t)z(t-\tau)+z'(t-r)-hQ(t-2\tau+\delta)\int_{t-\tau+\delta}^{t}[-z'(s-\delta)]\mathrm{d}s\\
&= -[\overline{P}(t)+hQ(t-2\tau+\delta)]z(t-\tau)+hQ(t-2\tau+\delta)z(t-\delta)+z'(t-r),
\end{aligned}$$

which implies that $-z(t)$ is a positive solution of the inequality (22). This is a contradiction and hence the proof is complete.

Example 2 Consider the following neutral equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)-(1+t^{-1})x(t-1)]+(t^{-\alpha}+(t-1)^{-\beta})x(t-2)-t^{-\beta}x(t-1)=0, \tag{23}$$

where $1<\alpha<3/2$ and $\beta>\alpha$. Here, we have eventually

$$R(t-\tau)=1+1/(t-2)\le 1+\alpha/(t-1),$$

and

$$[\overline{P}(t)]^{-1}\overline{P}(t-r)=(1+1/(t-1))^{\alpha}\ge 1+\alpha/(t-1).$$

It is clear that for sufficiently large t, (20) and (21) hold for $h=1$. Consider the following neutral inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}[y(t)-y(t-1)]+[t^{-\alpha}+(t-3)^{-\beta}]y(t-2)-(t-3)^{-\beta}y(t-1)\le 0,$$

which, clearly, satisfies all conditions of Corollary 1 . By Theorem 2 and Corol lary 1 every solution of equation (23) oscillates.

Remark 1 For equation (23), $h_1 = 1$ in (†), which implies that $h_2 = 0$ in (††). Since (23) does not satisfy (†) and (††) with $h_1 = 1$ and $h_2 = 0$, Theorem B[6] can't be applied to (23).

Remark 2 Example 2 shows that Theorem 2, combining with Theorem 1, can be applied to some neutral equations that don't satisfy the condition $R(t) \leq 1$.

References:

[1] CHUANXI Q, LADAS G. *Oscillation in differential equations with positive and negative coefficients* [J]. Canad. Math. Bull., 1990, **33**: 442–450.

[2] ERBE L H, KONG Qing-kai, ZHANG B G. *Oscillation Theory for Functional Differential Equations* [M]. Dekker, New York, 1995.

[3] GYORI I, LADAS G. *Oscillation Theory of Delay Differential Equations with Applications* [M]. Oxford Univ. Press(Clarendon), London/New York, 1991.

[4] RUAN S G. *Oscillations for first order neutral differential equations with variable coefficients* [J]. Bull. Austral. Math. Soc., 1991, **43**: 147–152.

[5] YU J S. *Neutral differential equations with positive and negative coefficients* [J]. Acta Math. Sinica, 1991, **34**: 517–523.

[6] YU J S, YAN J. *Oscillation in first order neutral differential equations with "Integrally small" coefficients* [J]. J. Math. Anal. Appl., 1994, **187**: 361–370.

具有"积分小"系数的中立型方程的振动性

张晓声[1]， 王 铎[2]

(1. 首都师范大学数学系，北京 100037; 2. 北京大学数学科学学院，北京 100871)

摘 要: 讨论了中立型方程

$$\frac{d}{dt}[x(t) - R(t)x(t-r)] + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0,$$

的振动性，其中 $P, Q, R \in C([t_0, \infty), R^+), r, \tau, \delta \in (0, \infty)$, 得到若干新结果.