Journal of Mathematical Research & Exposition
Vol.22, No.4, 575–582, November, 2002

Some Generalizations for Inequalities of Hua-Wang Type *

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Abstract: Some chains of inequalities of Hua-Wang type are established. These interesting results are the generalizations of some known inequalities. One of the new methods establishing inequalities is based on Sándor's good idea; another is applying the characteristics of nonlinear positive functionals.

Key words: inequality; generalization; method.

Classification: AMS(2000) 26D15/CLC O178.1

Document code: A **Article ID:** 1000-341X(2002)04-0575-08

1. Introducation and notations

In order to study the condition of positive solvability, Lo-Keng Hua[1] gave a result as follows: If $\delta > 0, \alpha > 0$,then

$$\alpha \cdot \Sigma x_i^2 + (\delta - \Sigma x_i)^2 \geq k_n \delta^2, \tag{1}$$

with equality if and only if

$$x_1 = \cdots = x_n = \delta/(n+\alpha), \tag{2}$$

where $k_n := \alpha(n+\alpha)^{-1}, x_1 \geq 0, \cdots, x_n \geq 0, x_1 + \cdots + x_n \leq \delta$.

As pointed out by L.-K.Hua and C.-L.Wang, (1) is a very interesting and useful inequality. For this reason, C.-L.Wang[2], W.-l.Wang[3] and C.E.M.Pearce and J.E.Pečarić[4] established several generalizations of (1). In Sections 2 and 3 of this paper we shall establish some interesting chains of inequalities, in both discrete and continuous versions, which are different from [2-4]. We may regard these inequalities as further generalizations of some results obtained recently in [2-3]. They are not only new, but also the true meaning from some mathematical and aesthetical points of view. In Theorem 1, a known inequality will be proved by means of three ways. It must be noticed that we will apply

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a very new method to prove Theorem 2 and the case (ii) of Theorem 1.This method is based on Sándor's idea[5] (also see [9]), namely, using the obvious fact

$$\sum \inf_{x\in E} f_i(x) \le \inf_{x\in E} \sum f_i(x) \tag{3}$$

to prove the inequalities. Another interesting method is applying the characteristics of nonlinear positive functionals[6] to establish the continuous versions of the discrete inequalities.

Let us show off some notations and symbols that we shall need:

$$X := (x_1, \cdots, x_n) \in \Re^n; \Re := \text{the field of real numbers};$$

$$\Re_+ := \{x|0 \le x < +\infty\}; \Re_{++} := \{x|0 < x < +\infty\};$$

$$\Omega := \{X|x_1 \ge 0, \cdots, x_n \ge 0, x_1 + \cdots + x_n \le \delta\}; N := \text{the set of natural numbers};$$

$$\Omega_1 := \{X|x_1 > 0, \cdots, x_n > 0, x_1 + \cdots + x_n \le \delta\}; \Omega^\circ := \text{the interior of } \Omega;$$

$$F_n(X,p) := F(p) := \alpha^{p-1} \cdot \sum x_i^p + (\delta - \sum x_i)^p; k_n := \alpha(n+\alpha)^{-1};$$

$$G(f,p) := G(p) := \alpha^{p-1} \int_0^T f^p \mathrm{d}t + (\delta - \int_0^T f \mathrm{d}t)^p; \sum := \sum_{i=1}^n; \prod := \prod_{i=1}^n.$$

Lemma *let* $X = (x_1, \cdots, x_n), W = (w_1, \cdots, w_n), x_i \ge 0, w_i > 0, i = 1, \cdots, n$*.Then for* $\alpha < \beta$, $M_\alpha(X,W) \le M_\beta(X,W)$, *where*

$$M_r := M_r(X,W) := \begin{cases} [(\sum w_i x_i^r)/(\sum w_i)]^{1/r}, & 0 < |r| < +\infty, \\ (\prod x_i^{w_i})^{1/(\sum w_i)}, & r = 0. \end{cases}$$

There is strict inequality unless the x_i are all equal or someone $x_i = 0$ and $r \le 0$. Replacing M_r by m_r, the above inequality also holds,where $w(t) > 0, f(t) \ge 0, t \in E$,and

$$m_r := m_r(f,w,E) := \begin{cases} [(\int_E w(t)(f(t))^r \mathrm{d}t)/(\int_E w(t)\mathrm{d}t)]^{1/r}, & 0 < |r| < +\infty, \\ \exp[(\int_E w(t) ln f(t)\mathrm{d}t)/(\int_E w(t)\mathrm{d}t)], & r = 0. \end{cases}$$

For details of this lemma, see [7,pp.4-6,24-25]. These results will be used.

2. Inequalities of discrete case

We first prove some known results [2-4] that will be used. For saving space, we only give a proof for each case in Theorem 1, although there are several proofs.

Theorem 1 *(i) Let positive real numbers* δ *and* α *be given. Then for* $p > 1$*, the inequality*

$$F_n(X,p) \ge k_n^{p-1}\delta^p \tag{4}$$

holds for all $X \in \Omega$*.*

(ii) If $p < 0$ *then (4) holds for all* $X \in \Omega_1$*.*

(iii) If $0 < p < 1$ then the reverse inequality in (4) holds for all $X \in \Omega$. In all cases equality obtains if and only if (2) holds.

Proof Case (i). For $p > 1$,choose $\Phi : \Omega \to \Re$ defined by $\Phi(X) := \alpha^{p-1} \cdot \sum x_i^p + (\delta - \Sigma x_i)^p$. It is not difficult to prove that Hessian matrix

$$[\Phi_{ij}]_n := \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \cdots & \cdots & \cdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{bmatrix}$$

is positive definite on Ω°,where $\Phi_{ij} = \partial^2 \Phi / \partial x_i \partial x_j$.In fact,from the given conditions we obtain that each principle minor of the above matrix

$$\det[\Phi_{ij}]_k = (p(p-1)\alpha^{p-1})^k (\Pi_{i=1}^k x_i)^{p-2} [1 + (\delta - \sum x_i)^{p-2} \cdot \alpha^{1-p} \cdot \sum_{i=1}^k x_i^{2-p}] > 0,$$

$$\forall X \in \Omega^\circ, (k = 1, 2, \cdots, n).$$

It follows from the above account that Φ is strictly convex on Ω°. We obtain $\Phi(X) \geq \Phi(X_\circ) = k_n^{p-1} \delta^p$,where $X_\circ := (\delta(n+\alpha)^{-1}, \cdots, \delta(n+\alpha^{-1}))$ is a unique critical point(see,e.g., [8,pp.103,123]). In other words, (4) has been proved.

Case(ii). For $p < 0$,choose the functions $f_i : \Re_{++} \to \Re$ defined by

$$f_i(x) = \begin{cases} \alpha x_i x - x^{p/(p-1)}, & i = 1, \cdots, n \\ \alpha(\delta - \sum x_i) x - \alpha x^{p/(p-1)}, & i = n+1. \end{cases}$$

Since

$$f_i'(x) = \begin{cases} \alpha x_i - [p/(p-1)] x^{1/(p-1)}, & i = 1, \cdots, n \\ \alpha(\delta - \sum x_i) - \alpha[p/(p-1)] x^{1/(p-1)}, & i = n+1; \end{cases}$$

$$f_i''(x) = \begin{cases} -[p/(p-1)^2] x^{(2-p)/(p-1)}, & i = 1, \cdots, n \\ -\alpha[p/(p-1)^2] x^{(2-p)/(p-1)}, & i = n+1, \end{cases}$$

from the given conditions we get $f_i''(x) > 0, i = 1, \cdots, n+1$, and f_i has minimum at

$$x_{i0} := \begin{cases} [\alpha(p-1)x_i/p]^{p-1}, & i = 1, \cdots, n \\ [(p-1)(\delta - \sum x_i)/p]^{p-1}, & i = n+1 \end{cases}$$

and its value is

$$f_i(x_{i0}) = \begin{cases} (\alpha x_i)^p [(\frac{p-1}{p})^{p-1} - (\frac{p-1}{p})^p], & i = 1, \cdots, n \\ \alpha(\delta - \sum x_i)^p [(\frac{p-1}{p})^{p-1} - (\frac{p-1}{p})^p], & i = n+1. \end{cases}$$

Adding the above functions, we have $f(x) := \sum_{i=1}^{n+1} f_i(x) = \alpha \delta x - (n+\alpha) x^{p/(p-1)}$. It is easy to prove that $f(x)$ has minimum value at $x_0 := (k_n \delta)^{p-1} (\frac{p-1}{p})^{p-1}$ and

$$f(x_0) = \alpha k_n^{p-1} \delta^p [(\frac{p-1}{p})^{p-1} - (\frac{p-1}{p})^p].$$

Using (3) we have

$$\alpha(\alpha^{p-1}\cdot\sum x_i^p+(\delta-\sum x_i)^p)[(\frac{p-1}{p})^{p-1}-(\frac{p-1}{p})^p] \le \alpha k_n^{p-1}\delta^p[(\frac{p-1}{p})^{p-1}-(\frac{p-1}{p})^p]. \quad (5)$$

From $[(p-1)/p]^{p-1}-[(p-1)/p]^p<0$,(5) is equivalent to (4).

Case (iii). For $0<p<1$, using Lemma we get

$$[\frac{\sum\alpha^{-1}(\alpha x_i)^p+(\delta-\sum x_i)^p}{n\alpha^{-1}+1}]^{1/p} \le \frac{\sum\alpha^{-1}(\alpha x_i)+(\delta-\sum x_i)}{n\alpha^{-1}+1},$$

or, simplifying, $\frac{\alpha^{p-1}\cdot\sum x_i^p+(\delta-\sum x_i)^p}{n\alpha^{-1}+1} \le (\frac{\delta}{n\alpha^{-1}+1})^p$, which is equivalent to the reverse inequality in (4),and equality is valid if and only if $\alpha x_1=\cdots=\alpha x_n=\delta-x_1-\cdots-x_n$, which is equivalent to (2).The proof of Theorem 1 is complete.

Remark 1 In [2-4] Theorem 1 was proved by means of dynamic programming, Schur-convexity and Jensen's inequality. The continuous counterpart of Theorem 1 can be naturally established (see [2][4]). However, we shall establish a more general chain of the inequalities in Theorems 3.

In Theorem 2 we shall use the symbol: $F(p)=\alpha^{p-1}\cdot\sum x_i^p+(\delta-\sum x_i)^p$.

Theorem 2 *Let positive real numbers δ and α be given.*

(i) If $p>\overline{N}+1(\overline{N}\in N)$,then for any $M\in N$, the inequalities

$$\begin{aligned}\cdots &\ge k_n^{-M/(p+M)}[F(p+M)]^{p/(p+M)} \ge \cdots \ge k_n^{-2/(p+2)}[F(p+2)]^{p/(p+2)}\\ &\ge k_n^{-1/(p+1)}[F(p+1)]^{p/(p+1)} \ge F(p)\\ &\ge k_n^{1/(p-1)}[F(p-1)]^{p/(p-1)} \ge k_n^{2/(p-2)}[F(p-2)]^{p/(p-2)}\\ &\ge \cdots \ge k_n^{\overline{N}/(p-\overline{N})}[F(p-\overline{N})]^{p/(p-\overline{N})} \ge k_n^{p-1}\delta^p. \end{aligned} \quad (6)$$

hold for all $X\in\Omega$.

(ii) If $p<-\overline{N}(\overline{N}\in N)$,then for any $M\in N$, the inequalities

$$\begin{aligned}\cdots &\ge k_n^{M/(p-M)}[F(p-M)]^{p/(p-M)} \ge \cdots \ge k_n^{2/(p-2)}[F(p-2)]^{p/(p-2)}\\ &\ge k_n^{1/(p-1)}[F(p-1)]^{p/(p-1)} \ge F(p)\\ &\ge k_n^{-1/(p+1)}[F(p+1)]^{p/(p+1)} \ge k_n^{-2/(p+2)}[F(p+2)]^{p/(p+2)}\\ &\ge \cdots \ge k_n^{-\overline{N}/(p+\overline{N})}[F(p+\overline{N})]^{p/(p+\overline{N})} \ge k_n^{p-1}\delta^p \end{aligned} \quad (7)$$

hold for all $X\in\Omega_1$.

(iii) If $0<p<1$,then for any $M\in N$, the inequalities

$$\begin{aligned}\cdots &\le k_n^{M/(p-M)}[F(p-M)]^{p/(p-M)} \le \cdots \le k_n^{2/(p-2)}[F(p-2)]^{p/(p-2)}\\ &\le k_n^{1/(p-1)}[F(p-1)]^{p/(p-1)} \le F(p) \le k_n^{p-1}\delta^p \end{aligned} \quad (8)$$

hold for all $X\in\Omega_1$.

Proof Case (i): For $p > \overline{N}+1$, choose $f_i : \Re_+ \to \Re$ defined by

$$f_i(x) := \begin{cases} \alpha^{-1}(x^p - p\alpha^{p-1}x_i^{p-1}x), & i = 1, \cdots, n \\ x^p - p(\delta - \sum x_i)^{p-1}x, & i = n+1. \end{cases} \tag{9}$$

It is not difficult to prove that

$$\inf_{x\in\Re_+} f_i(x) = \begin{cases} f_i(\alpha x_i) = (1-p)\alpha^{p-1}x_i^p, & i = 1, \cdots, n \\ f_i(\delta - \sum x_i) = (1-p)(\delta - \sum x_i)^p, & i = n+1. \end{cases}$$

Let $f := \sum_{i=1}^{n+1} f_i$. Then $f(x) = (n\alpha^{-1}+1)x^p - p[\alpha^{p-2}\cdot\sum x_i^{p-1} + (\delta - \sum x_i)^{p-1}]x$. Similarly we obtain

$$\inf_{x\in\Re_+} f(x) = f(x_0) = (1-p)k_n^{1/(p-1)}[\alpha^{p-2}\cdot\sum x_i^{p-1} + (\delta - \sum x_i)^{p-1}]^{p/(p-1)},$$

where $x_0 := k_n^{1/(p-1)}[\alpha^{p-2}\cdot\sum x_i^{p-1} + (\delta - \sum x_i)^{p-1}]^{1/(p-1)}$. Using (3) we have

$$\begin{aligned} &(1-p)[\alpha^{p-1}\cdot\sum x_i^p + (\delta - \sum x_i)^p] \\ &\le (1-p)k_n^{1/(p-1)}[\alpha^{p-2}\cdot\sum x_i^{p-1} + (\delta - \sum x_i)^{p-1}]^{p/(p-1)} \end{aligned} \tag{10}$$

Dividing by $(1-p)$ from both of (10) and using the above symbol gives

$$F(p) \ge k_n^{1/(p-1)}[F(p-1)]^{p/(p-1)}. \tag{11}$$

Note that (11) can be used successively:

$$\begin{aligned} F(p) &\ge k_n^{1/(p-1)}[F(p-1)]^{p/(p-1)} \\ &\ge k_n^{1/(p-1)}[k_n^{1/(p-2)}(F(p-2))^{(p-1)/(p-2)}]^{p/(p-1)} \\ &= k_n^{2/(p-2)}[F(p-2)]^{p/(p-2)} \\ &\ge \cdots \ge k_n^{(\overline{N}-1)/(p-\overline{N}+1)}[F(p-\overline{N}+1)]^{p/(p-\overline{N}+1)} \\ &\ge k_n^{(\overline{N}-1)/(p-\overline{N}+1)}[k_n^{1/(p-\overline{N})}(F(p-\overline{N}))^{(p-\overline{N}+1)/(p-\overline{N})}]^{p/(p-\overline{N}+1)} \\ &= k_n^{\overline{N}/(p-\overline{N})}[F(p-\overline{N})]^{p/(p-\overline{N})} \\ &\ge k_n^{\overline{N}/(p-\overline{N})}[k_n^{p-\overline{N}-1}\delta^{p-\overline{N}}]^{p/(p-\overline{N})} = k_n^{p-1}\delta^p. \end{aligned} \tag{12}$$

Here the last step used (4) in Theorem 1. On the other hand, replacing p by $p+1$ in (11) we have $k_n^{1/p}[F(p)]^{(p+1)/p} \le F(p+1)$. This inequality may be written equivalently as

$$F(p) \le k_n^{-1/(p+1)}[F(p+1)]^{p/(p+1)}. \tag{13}$$

(13) can also be used successively:

$$\begin{aligned} F(p) &\le k_n^{-1/(p+1)}[F(p+1)]^{p/(p+1)} \\ &\le k_n^{-1/(p+1)}[k_n^{-1/(p+2)}(F(p+2))^{(p+1)/(p+2)}]^{p/(p+1)} \\ &= k_n^{-2/(p+2)}[F(p+2)]^{p/(p+2)} \le \cdots \\ &\le k_n^{-(M-1)/(p+M-1)}[F(p+M-1)]^{p/(p+M-1)} \\ &\le k_n^{-(M-1)/(p+M-1)}[k_n^{-1/(p+M)}(F(p+M))^{(p+M-1)/(p+M)}]^{p/(p+M-1)} \\ &= k_n^{-M/(p+M)}[F(p+M)]^{p/(p+M)} \le \cdots. \end{aligned} \tag{14}$$

Combining (14) with (12),the desired (6) can be obtained.

Case(ii): For $p < -N$, choose $f_i : \Re_{++} \to \Re$ defined by (9). Since we can also establish (7) by the above similar method, therefore we omit its proof.

Case (iii): For $0 < p < 1$, from Lemma for any $M \in N$ we have

$$[\frac{\sum \alpha^{-1}(\alpha x_i)^{p-j} + (\delta - \sum x_i)^{p-j}}{n\alpha^{-1}+1}]^{1/(p-j)} \le [\frac{\sum \alpha^{-1}(\alpha x_i)^{p-j+1} + (\delta - \sum x_i)^{p-j+1}}{n\alpha^{-1}+1}]^{1/(p-j+1)},$$

where $j = 1, 2, \cdots, M, M+1, \cdots$.

Raising both sides to the pth power, and dividing k_n, we have

$$k_n^{j/(p-j)}[F(p-j)]^{p/(p-j)} \le k_n^{(j-1)/(p-j+1)}[F(p-j+1)]^{p/(p-j+1)},$$

where $j = 1, 2, \cdots, M, M+1, \cdots$

Combining these inequalities with (iii) of Theorem 1, we can obtain the desired (8). This completes the proof of Theorem 2.

Remark 2 As a special case, we have Theorem 1. For example, letting $p - \overline{N} = \beta(> 1)$in the last inequality of (6), we can obtain $F(\beta) \ge k_n^{\beta-1}\delta^{\beta}$.

3. Inequalities of continuous case

We shall use the following symbols displayed in Section 1: Let $E = [0, T], w(t) = 1$, then

$$G(f,p) := G(p) := \alpha^{p-1}\int_0^T f^p \mathrm{d}t + (\delta - \int_0^T f\mathrm{d}t)^p; \quad k_T := \alpha(T+\alpha)^{-1};$$
$$m_p := m_p(\alpha f, w, E) := [T^{-1}\cdot\int_0^T(\alpha f)^p]^{1/p}; \quad m := m_1 := T^{-1}\cdot\int_0^T \alpha f\mathrm{d}t.$$

Theorem 3 *Let positive real numbers δ and α be given.*

(i) If $p > \overline{N}+1(\overline{N} \in N)$, for any $M \in N$, the inequalities

$$\begin{aligned}\cdots &\ge k_T^{-M/(p+M)}[G(p+M)]^{p/(p+M)} \ge \cdots \ge k_T^{-2/(p+2)}[G(p+2)]^{p/(p+2)}\\ &\ge k_T^{-1/(p+1)}[G(p+1)]^{p/(p+1)} \ge G(p)\\ &\ge k_T^{1/(p-1)}[G(p-1)]^{p/(p-1)}\\ &\ge k_T^{2/(p-2)}[G(p-2)]^{p/(p-2)} \ge \cdots\\ &\ge k_T^{\overline{N}/(p-\overline{N})}[G(p-\overline{N})]^{p/(p-\overline{N})} \ge k_T^{p-1}\delta^p \end{aligned} \quad (15)$$

hold for any positive integrable function f on $0 \le t < T < +\infty$ with $\int_0^T f\mathrm{d}t \le \delta$.

(ii) If $p < -\overline{N}(\overline{N} \in N)$,then for any $M \in N$, the inequalities

$$\begin{aligned}\cdots &\ge k_T^{M/(p-M)}[G(p-M)]^{p/(p-M)} \ge \cdots \ge k_T^{2/(p-2)}[G(p-2)]^{p/(p-2)}\\ &\ge k_T^{1/(p-1)}[G(p-1)]^{p/(p-1)} \ge G(p)\\ &\ge k_T^{-1/(p+1)}[G(p+1)]^{p/(p+1)} \ge k_T^{-2/(p+2)}[G(p+2)]^{p/(p+2)} \ge \cdots\\ &\ge k_T^{-\overline{N}/(p+\overline{N})}[G(p+\overline{N})]^{p/(p+\overline{N})} \ge k_T^{p-1}\delta^p \end{aligned} \quad (16)$$

hold for any positive integrable function f *on* $0 \le t < T < +\infty$ *with* $\int_0^T f dt \le \delta$, *where* $0 < \beta \le f(x)$, *and* β *is given.*

(iii) If $0 < p < 1$, *then for any* $M \in N$, *the inequalities*

$$\cdots \le k_T^{M/(p-M)}[G(p-M)]^{p/(p-M)} \le \cdots \le k_T^{2/(p-2)}[G(p-2)]^{p/(p-2)}$$
$$\le k_T^{1/(p-1)}[G(p-1)]^{p/(p-1)} \le G(p) \le k_T^{p-1}\delta^p \tag{17}$$

hold for any positive integrable function f *on* $0 \le t < T < +\infty$ *with* $\int_0^T f\mathrm{d}t \le \delta$, *where* $0 < \beta \le f(x)$, *and* β *is given.*

Proof Case (i): First we prove that the following inequalities hold

$$k_T^{j/(p-j)}[G(p-j)]^{p/(p-j)} \ge k_T^{(j+1)/(p-j-1)}[G(p-j-1)]^{p/(p-j-1)}, \tag{18}$$

where $j = \cdots, -M, \cdots, -2, -1, 0, 1, 2, \cdots, \overline{N}-1$.

The above inequalities can be written equivalently as

$$[\frac{T \cdot m_{p-j}^{p-j} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-j}}{T+\alpha}]^{1/(p-j)} \ge [\frac{T \cdot m_{p-j-1}^{p-j-1} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-j-1}}{T+\alpha}]^{1/(p-j-1)},$$

where $j = \cdots, -M, \cdots, -2, -1, 0, 1, 2, \cdots, \overline{N}-1$, and $m_r^r = T^{-1} \cdot \int_0^T (\alpha f)^r \mathrm{d}t$. From Lemma we have $m_{p-j} \ge m_{p-j-1}$ and so that

$$[\frac{T \cdot m_{p-j}^{p-j} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-j}}{T+\alpha}]^{1/(p-j)} \ge [\frac{T \cdot m_{p-j-1}^{p-j} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-j}}{T+\alpha}]^{1/(p-j)}$$
$$\ge [\frac{T \cdot m_{p-j-1}^{p-j-1} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-j-1}}{T+\alpha}]^{1/(p-j-1)}.$$

It follows from the above inequalities that (18) hold for $j = \cdots, -M, \cdots, -2, -1, 0, 1, 2, \cdots$ 1. Now we prove the last inequality in (15), i.e.,

$$k_T^{\overline{N}/(p-\overline{N})}[\alpha^{p-\overline{N}-1}\int_0^T f^{p-\overline{N}}\mathrm{d}t + (\delta - \int_0^T f\mathrm{d}t)^{p-\overline{N}}]^{p/(p-\overline{N})} \ge k_T^{p-1}\delta^p. \tag{19}$$

In order to display the methods establishing inequalities,we shall use two methods to prove (19) as follows:

First method. Using Lemma for $p - \overline{N} > 1$, we obtain

$$[\frac{T \cdot m_{p-\overline{N}}^{p-\overline{N}} + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)^{p-\overline{N}}}{T+\alpha}]^{1/(p-\overline{N})} \ge \frac{T \cdot m + \alpha \cdot (\delta - \int_0^T f\mathrm{d}t)}{T+\alpha} = \frac{\alpha\delta}{T+\alpha} = k_T\delta.$$

It is easy to see that this inequality is equivalent to (19).

Second method. We can treat G as a nonlinear positive functional of integrable functions. Let $G := G(f, p-\overline{N}) = \alpha^{p-\overline{N}-1}\int_0^T f^{p-\overline{N}}\mathrm{d}t + (\delta - \int_0^T f\mathrm{d}t)^{p-\overline{N}}$. Working in a similar way as [2,p.349], we can obtain that the first and second Gateaux differentials of G are the following, respectively,

$$\sigma G = (p - \overline{N})[\alpha^{p-\overline{N}-1} \int_0^T f^{p-\overline{N}-1}\mathrm{d}t - T(\delta - \int_0^T f\mathrm{d}t)^{p-\overline{N}-1}],$$

$$\sigma^2 G = (p - \overline{N})(p - \overline{N} - 1)[\alpha^{p-\overline{N}-1} \int_0^T f^{p-\overline{N}-2}\mathrm{d}t + T^2(\delta - \int_0^T f\mathrm{d}t)^{p-\overline{N}-2}],$$

Since $\sigma G = 0$ if and only if $f = f_0 = \delta(T+\alpha)^{-1}$, and $\sigma^2 G > 0$, therefore $G(f) := G(f, p - \overline{N})$ has a unique minimum value $G(f_0) = k_T^{p-\overline{N}-1}\delta^{p-\overline{N}}$. In other words, we have

$$G(f, p - \overline{N}) \geq k_T^{p-\overline{N}-1}\delta^{p-\overline{N}},$$

which is equivalent to (19). Combining (18) with (19), inequalities (15) are obtained.

It is not difficult to prove (16) and (17) by means of the similar methods above, so we omit them.

Acknowledgments The authors are indebted to the unknown referees for pointing out a major error and proposing helpful suggestions.

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华罗庚 - 王中烈不等式的一些推广

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摘 要: 在数学上和美学上都有意义的一些不等式链，是已知的一类华罗庚 - 王中烈型不等式的推广. 在建立不等式的方法中有两种是近些年出现的：其一是基于最近 Sándor 的恰当想法；其二是利用非线性正泛函的特性. 本文展示了具有新形式的不等式，还展示了建立它们的若干方法.