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Existence of Solutions for Forward-Backward Stochastic Differential Equations with Jumps and Non-Lipschitzian Coefficients *

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Abstract: This paper studies forward-backward differential equations with Poisson jumps and with stopping time as termination. Under some weak monotonicity conditions and for non-Lipschitzian coefficients, the existence and uniqueness of solutions are proved via a purely probabilistic approach, while a priori estimate is given. Here, we allow the forward equation to be degenerate.

Key words: Forward-backward stochastic differential equations; Unbounded stopping time; Non-Lipschitzian coefficients; Priori estimate.

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1. Introduction

Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion, $(k(t))_{t\geq 0}$ a Poisson point process taking value in a measurable space $(Z, \mathcal{B}(Z))$, $N_k(\mathrm{d}s, \mathrm{d}z)$ a Poisson counting measure defined by $k(\cdot)$ with compensator $\Pi(\mathrm{d}z)\mathrm{d}s$, $\tilde{N}_k(\mathrm{d}s, \mathrm{d}z)$ the martingale measure such that $\tilde{N}_k(\mathrm{d}s, \mathrm{d}z) = N_k(\mathrm{d}s, \mathrm{d}z) - \Pi(\mathrm{d}z)\mathrm{d}s$, and $\Pi(\cdot)$ a σ-finite measure on $\mathcal{B}(Z)$.

Let$(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration denoted by $\mathcal{F}_t = \sigma[W_s; s \leq t] \vee \sigma[N_k(A, (0, s]); s \leq t, A \in \mathcal{B}(Z)] \vee N$, where N is the all P-null sets. We consider the following forward-backward stochastic differential equations with jumps

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(FBSDEJs):

$$
\begin{aligned}
x_t =& x_0 + \int_0^{t\wedge\tau} b(s, x_s, y_s, q_s, p_s, \omega)\mathrm{d}s + \int_0^{t\wedge\tau} \sigma(s, x_s, y_s, q_s, p_s, \omega)\mathrm{d}W_s + \\
& \int_0^{t\wedge\tau} \int_Z c(s, x_{s-}, y_{s-}, q_s, p_s, z, \omega)\widetilde{N}_k(\mathrm{d}s, \mathrm{d}z), \qquad (1.1)\\
y_t =& \psi(x_\tau) + \int_{t\wedge\tau}^{\tau} h(s, x_s, y_s, q_s, p_s, \omega)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} q_s \mathrm{d}W_s - \\
& \int_{t\wedge\tau}^{\tau} \int_Z p_s(z)\widetilde{N}_k(\mathrm{d}z, \mathrm{d}s), \qquad (1.2)
\end{aligned}
$$

where (x, y, q, p) takes values in $(R^n \times R^m \times R^{m\times d} \times R^m)$, $x_0 \in R^n$, and τ is a stopping time and takes values in $[0, +\infty]$. The functions b, h, σ are jointly measurable and $\mathcal{F}_t$-adapted; c is jointly measurable and $\mathcal{F}_t$-predictable; and ψ is jointly measurable with respect to $\mathcal{B}(R^n) \times \mathcal{F}_\tau$.

$$
\begin{aligned}
& b : [0, \infty) \times R^n \times R^m \times R^{m\times d} \times L^2_{\Pi(\cdot)}(R^m) \times \Omega \longrightarrow R^n, \\
& h : [0, \infty) \times R^n \times R^m \times R^{m\times d} \times L^2_{\Pi(\cdot)}(R^m) \times \Omega \longrightarrow R^m, \\
& \sigma : [0, \infty) \times R^n \times R^m \times R^{m\times d} \times L^2_{\Pi(\cdot)}(R^m) \times \Omega \longrightarrow R^{n\times d}, \\
& c : [0, \infty) \times R^n \times R^m \times R^{m\times d} \times L^2_{\Pi(\cdot)}(R^m) \times Z \times \Omega \longrightarrow R^n, \\
& \psi : \Omega \times R^n \longrightarrow R^m,
\end{aligned}
$$

where $L^2_{\Pi(\cdot)}(R^m)$ will be given in next section.

FBSDEs with Brownian motion were first introduced by Antonelli[1]. In his work, he obtained a local existence and uniqueness result, where the coefficients satisfy Lipschitz conditions and are independent of the variable q. Due to broad applications to stochastic optimal control, mathematical economics and mathematical finance, there have appeared lots of results on FBSDEs. To our knowledge, there are two main methods to study FBSDEs. The first one is purely probabilistic method, which was given by Hu and Peng[2] and Peng and Wu[5], who apply Itô's formulas and then construct a contractive mapping to solve FBSDEs under a monotonicity condition and Lipschitz condition. Specially, in [5] the authors considered two cases: same dimension and different dimension. Similarly, Peng and Shi[6] investigated a class of system of infinite horizon FBSDEs. Under some monotonicity assumptions and terminal value of solution for BSDE being zero, the existence, uniqueness, and comparison theorem of FBSDEs are given. Pardoux and Tang[4] have also studied, under some natural monotonicity conditions, existence and uniqueness, a prior estimate, and established the connection with qusilinear parabolic PDEs, where the proof for existence and uniqueness of solution was based on the fixed point theorem by using an equivalent norm, while some coefficients are not Lipschitz continuous. The second was given by Ma, Protter and Yong[3], via a partial differential equation approach. But this method needs the forward equation to be non-degenerate and the coefficients not to be random. Tang and Li[10] initially applied the idea of Peng to get the first result on adapted solution to a BSDE with Poisson jumps and with Lipschitzian coefficients . And then SiTu[7] discussed the existence and uniqueness of solutions for BSDE with jumps with non-Lipschitzian coefficients. Wu[11] also studied FBSDEs with jumps and Lipschitzian coefficients. In particular, SiTu[8] have extensively researched on BSDEs and FBSDEs with Poisson jumps for an arbitrary time duration. This paper considers extended FBSDEs, i.e. forward-backward stochastic differential equation with Poisson jumps (in short FBSDEJs). Moreover, the time duration is a stopping time which is unbounded and can take infinite value.

The paper is organized as follows. In Section 2, we present some notations and the assumptions that coefficients and terminal function satisfy, then give a priori estimate; Section 3 is devoted to the proof of the existence and uniqueness of solutions to FBSDEJs under non-Lipschitz conditions.

2. Preliminary: notations and a priori estimate

In this section, we will mainly present a priori estimate, which is very important for proving existence and uniqueness of solutions. We introduce following notations:

$$\begin{aligned}
S^2_{\mathcal{F}_t}(R^n) =&\{v(t,\omega) : v(t,\omega) \text{ is } R^n \text{ valued, } \mathcal{F}_t-\text{adapted such that } \|v\cdot\|_1^2 \\
=&E\sup_{0\le t\le\tau}|v(t,\omega)|^2<\infty\};\\
L^2_{\mathcal{F}_t}(R^n) =&\{v(t,\omega) : v(t,\omega) \text{ is } R^n \text{ valued, } \mathcal{F}_t-\text{adapted such that } \|v\cdot\|_2^2 \\
=&E\int_0^\tau |v(t,\omega)|^2<\infty\};\\
F^2_{\mathcal{F}_t}(R^n) =&\{u(t,z,\omega) : u(t,z,\omega) \text{ is } R^n \text{ valued, } \mathcal{F}_t-\text{predictable such that}\\
&\|u\cdot\|_3^2 = E\int_0^\tau\int_Z |u(t,z,\omega)|^2\Pi(\mathrm{d}z)\mathrm{d}t<\infty\};\\
L^2_{\Pi(\cdot)}(R^m) =&\{u(z) : u(z) \text{ is a } \mathcal{B}(Z) \text{ measurable, } R^m-\text{valued such that } \|u\|\\
=&(\int_Z |u(z)|^2\Pi(\mathrm{d}z))^{\frac{1}{2}}<\infty\};\\
M^2 =&\{\xi : \xi \text{ is an } \mathcal{F}_\tau \text{ measurable, such that } E|\xi|^2<\infty\}.
\end{aligned}$$

Obviously, the space $S^2_{\mathcal{F}_t}(R^n)\times S^2_{\mathcal{F}_t}(R^m)\times L^2_{\mathcal{F}_t}(R^{m\times d})\times F^2_{\mathcal{F}_t}(R^m)$ is a Banach space.

For notation simplification, here we only consider the case $n=m$. Similarly, we can use the techniques in [5] to deal with the case $n\neq m$. Moreover, all results are remain correct when $k(\cdot)$ is a d_1- dimensional Poisson point process(c and q have a proper dimension). We will use the following notations:

$$\begin{aligned}
u =&(x,y,q,p),\quad A(t,u,\omega)=(-h(t,u,\omega),b(t,u,\omega),\sigma(t,u,\omega),c(t,u,\cdot,\omega)),\\
\langle u,A\rangle =&u\cdot A=-\langle x,h\rangle+\langle y,b\rangle+\langle q,\sigma\rangle+\ll p,c\gg;\\
\ll p,c\gg =&\int_Z p_t(z)\cdot c(t,u,z)\Pi(\mathrm{d}z),
\end{aligned}$$

where $\langle a,b\rangle=a\cdot b$ is the usual inner product with Euclidean norm in R^n and $R^{n\times d}$. All the equalities and inequalities mentioned in this paper are in the sense of $dt\times dP$ almost surely on $[0,\infty)\times\Omega$.

Definition 2.1 *(x_t, y_t, q_t, p_t) is called an adapted solution of (1.1) and (1.2), if and only if*

1) $(x_t, y_t, q_t, p_t) \in S^2_{\mathcal{F}_t}(R^n) \times S^2_{\mathcal{F}_t}(R^n) \times L^2_{\mathcal{F}_t}(R^{n\times d}) \times F^2_{\mathcal{F}_t}(R^n)$;

2) (x_t, y_t, q_t, p_t) satisfies (1.1) and (1.2).

The following assumptions are necessary:

(H1) The functions b, h, σ, c are continuous with respect to $(x, y, q, p) \in S^2_{\mathcal{F}_t}(R^n) \times S^2_{\mathcal{F}_t}(R^n) \times L^2_{\mathcal{F}_t}(R^{n\times d}) \times F^2_{\mathcal{F}_t}(R^n)$.

(H2) The process $A(t, x, y, q, p, \omega) = A_1(t, x, y, q, p, \omega) + A_2(t, x, y, q, p, \omega)$, moreover, $|h_1(t, x, y, q, p, \omega)| + |b_1(t, x, y, q, p, \omega)| \le u_1(t)$. Furthermore, for any $u_i = (x_i, y_i, q_i, p_i) \in R^{n+n+n\times d} \times L^2_{\Pi(\cdot)}(R^n), i = 1, 2$, the following hold:

$$\begin{aligned}
&|h_1(t, x, y, q_1, p_1, \omega) - h_1(t, x, y, q_2, p_2, \omega)| + |b_1(t, x, y, q_1, p_1, \omega) - b_1(t, x, y, q_2, p_2, \omega)| \\
&\quad \le u_2(t)[|q_1 - q_2| + \|p_1 - p_2\|] \\
&|\sigma_1(t, u_1, \omega) - \sigma_1(t, u_2, \omega)| + \|c_1(t, u_1, \cdot, \omega) - c_1(t, u_2, \cdot, \omega)\| \\
&\quad \le u_1(t)[|x_1 - x_2| + |y_1 - y_2|] + u_2(t)[|q_1 - q_2| + \|p_1 - p_2\|]
\end{aligned}$$

$$|A_2(t, u_1, \omega) - A_2(t, u_2, \omega)| \le u_1(t)[|x_1 - x_2| + |y_1 - y_2|] + u_2(t)[|q_1 - q_2| + \|p_1 - p_2\|]$$

where $u_1(t)$ and $u_2(t)$ are strictly positive deterministic functions and satisfy $\int_0^\infty (u_1(t) + u_2(t))\mathrm{d}t < \infty$. Moreover, there exists a positive constant C such that $u_i(t) \le C, i = 1, 2$.

(H3) Function ψ is uniformly Lipschitz continuous, that is, $|\psi(x_1) - \psi(x_2)| \le C|x_1 - x_2|$.

(H4) For function $\psi(\cdot)$, $\psi(0) \in M^2$. Moreover,

$$\begin{aligned}
&E(\int_0^T |h_2(s, 0, 0, 0, 0, \omega)|\mathrm{d}s)^2 + E(\int_0^T |b_2(s, 0, 0, 0, 0, \omega)|\mathrm{d}s)^2 + \\
&E\int_0^T |\sigma(s, 0, 0, 0, 0, \omega)|^2\mathrm{d}s + E\int_0^T \|c(s, 0, 0, 0, 0, \cdot, \omega)\|^2\mathrm{d}s = L_0 < \infty.
\end{aligned}$$

(H5) For any $u_i = (x_i, y_i, q_i, p_i)$, $i = 1, 2$, the following inequalities hold:

$$\begin{aligned}
\langle A(t, u_1) - A(t, u_2,), u_1 - u_2\rangle \le &- \beta_1 u_1(t)|x_1 - x_2|^2 - \beta_2 u_1(t)|y_1 - y_2|^2 - \\
&\beta_3 u_2(t)[|q_1 - q_2|^2 + \|p_1 - p_2\|^2] \\
\langle \psi(x_1) - \psi(x_2), x_1 - x_2\rangle \ge &\beta_4 |x_1 - x_2|^2,
\end{aligned}$$

where $\beta_i \ge 0, 1 \le i \le 4$ are constants, and satisfy one of the following conditions:

(1) β_1, $\beta_4 > 0$ and

$$\begin{aligned}
&\langle y_1 - y_2, h_1(t, x_1, y_1, q, p) - h_1(t, x_2, y_2, q, p)\rangle \\
&\quad \le u_1(t)\rho(|y_1 - y_2|^2) + u_1(t)|y_1 - y_2||x_1 - x_2|.
\end{aligned} \tag{2.1}$$

(2) β_2, $\beta_3 > 0$ and

$$\begin{aligned}
&\langle x_1 - x_2, b_1(t, x_1, y_1, q, p) - b_1(t, x_2, y_2, q, p)\rangle \\
&\quad \le u_1(t)\rho(|x_1 - x_2|^2) + u_1(t)|x_1 - x_2||y_1 - y_2|.
\end{aligned} \tag{2.2}$$

Lemma 2.2 *We make assumptions (H1)–(H4). (H5) is satisfied except (2.1) and (2.2). If (x_t, y_t, q_t, p_t) solves (1.1) and (1.2), then*

$$\|x\cdot\|_1^2 + \|y\cdot\|_1^2 + \|q\cdot\|_2^2 + \|p\cdot\|_3^2 \le C_0 < \infty.$$

where C_0 is a constant depending on $C, \beta_i, i = 1, 2, 3, 4, \int_0^\infty (u_1(t) + u_2(t))dt, L_0$, and $E|\psi(0)|^2$ only.

Proof Applying Itô's formula to $|y_t|^2$, we have

$$\begin{aligned}|y_t|^2 &+ \int_{t\wedge\tau}^{\tau} |q_s|^2 \mathrm{d}s + \int_{t\wedge\tau}^{\tau} \|p_s\|^2 \mathrm{d}s \\ &= |\psi(x_\tau)|^2 + \int_{t\wedge\tau}^{\tau} 2x_s\cdot h(s, x_s, y_s, q_s, p_s)\mathrm{d}s + \int_{t\wedge\tau}^{\tau} \mathrm{d}M_s\end{aligned}$$

where M_t is an $\mathcal{F}_t$-martingale. From this and Yang's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}&E|y_t|^2 + \frac{1}{2}E\int_{t\wedge\tau}^{\tau} |q_s|^2\mathrm{d}s + \frac{1}{2}E\int_{t\wedge\tau}^{\tau}\|p_s\|^2\mathrm{d}s \le 2C^2E|x_\tau|^2 + \\ &2E|\psi(0)|^2 + E\int_{t\wedge\tau}^{\tau}(4u_1(s) + 4u_2^2(s))|y_s|^2\mathrm{d}s + E\int_{t\wedge\tau}^{\tau} u_1(s)|x_s|^2\mathrm{d}s + \\ &\varepsilon\|y\cdot\|_1^2 + \varepsilon^{-1}E(\int_0^\tau |h_2(s, 0, 0, 0, 0)|\mathrm{d}s)^2 + \int_0^\infty u_1(s)\mathrm{d}s.\end{aligned}$$

Hence by the Gronwall's inequality

$$\sup_{0\le t\le\tau} E|y_t|^2 + E\int_0^\tau |q_s|^2\mathrm{d}s + E\int_0^\tau \|p_s\|^2\mathrm{d}s \le K_\varepsilon + KE|x_\tau|^2 + KE\int_0^\tau u_1(s)|x_s|^2\mathrm{d}s + K\varepsilon\|y\cdot\|_1^2.$$

Note that

$$\begin{aligned}|y_t|^2 =& |y_0|^2 - \int_0^t 2y_s\cdot h(s, u_s)\mathrm{d}s + \int_0^t 2y_s\cdot q_s\mathrm{d}W_s + \\ &\int_0^t\int_Z 2y_s\cdot p_s(z)\tilde{N}_k(\mathrm{d}z, \mathrm{d}s) + \int_0^t |q_s|^2\mathrm{d}s + \int_0^t\int_Z |p_s(z)|^2 N_k(\mathrm{d}z, \mathrm{d}s).\end{aligned}$$

From the martingale inequality, it follows that

$$\begin{aligned}\|y\cdot\|_1^2 \le& |y_0|^2 + E\int_0^\tau 2|y_s||h(s, u_s)|\mathrm{d}s + \|q\cdot\|^2 + \|p\cdot\|^2 + \\ &2\tilde{C}E(\int_o^\tau |y_s|^2|q_s|^2\mathrm{d}s)^{\frac{1}{2}} + 2\tilde{C}E(\int_o^\tau |y_s|^2\|p_s\|^2\mathrm{d}s)^{\frac{1}{2}} \\ \le& |y_0|^2 + E\int_0^\infty (4u_1(s) + 4u_2^2(s))|y_s|^2\mathrm{d}s + E\int_0^\tau u_1(s)|x_s|^2\mathrm{d}s + \\ &\frac{3}{4}\|y\cdot\|_1^2 + 4(\int_0^\tau |h_2(s, 0, 0, 0, 0)|\mathrm{d}s)^2 + \tilde{C}_*(\|q\cdot\|_2^2 + \|p\cdot\|_3^2),\end{aligned}$$

thus, by what we have just proved, taking a small enough $\varepsilon > 0$, we have

$$\|y\cdot\|^2 \le K + KE\int_0^\tau u_1(s)|x_s|^2\mathrm{d}s + KE|x_\tau|^2.$$

This leads to, in particular

$$\|q\cdot\|^2+\|p\cdot\|^2\le K+KE\int_0^\tau u_1(s)|x_s|^2\mathrm{d}s+KE|x_\tau|^2.$$

Applying Itô's formula to $\langle x_s,y_s\rangle$, we have

$$\begin{aligned}
&\beta_4E|x_\tau|^2+E\int_0^\tau(\beta_1u_1(s)|x_s|^2+\beta_2u_1(s)|y_s|^2+\beta_3u_2(|q_s|^2+\|p_s\|^2))\mathrm{d}s\\
&\le|x_0||y_0|+E|\psi(0)||x_\tau|+\varepsilon(\|x\cdot\|_1^2+\|y\cdot\|_1^2+\|q\cdot\|_2^2+\|p\cdot\|_3^2)+\\
&E\int_0^\tau u_1(s)|x_s|\mathrm{d}s+E\int_0^\tau u_1(s)|y_s|\mathrm{d}s+\\
&K_\varepsilon[E(\int_0^\tau|h_2(s,0,0,0,0)|\mathrm{d}s)^2+E(\int_0^\tau|b_2(s,0,0,0,0)|\mathrm{d}s)^2+\\
&E\int_0^\tau(|\sigma(s,0,0,0,0)|^2+\|c(s,0,0,0,0,\cdot)\|^2)\mathrm{d}s].
\end{aligned}$$

For the case β_1, $\beta_4>0$, the inequality

$$\begin{aligned}
&E|x_\tau|^2+E\int_0^\tau u_1(s)|x_s|^2\mathrm{d}s\\
&\le K_\varepsilon+K\varepsilon(\|x\cdot\|_1^2+\|y\cdot\|_1^2+\|q\cdot\|_2^2+\|p\cdot\|_3^2)
\end{aligned}$$

follows, and thus

$$\|y\cdot\|_1^2+\|q\cdot\|_2^2+\|p\cdot\|_3^2\le K_\varepsilon+K\varepsilon\|x\cdot\|_1^2,$$

where $K_\varepsilon>0$ and K can be different in above and below inequalities. Analogously, for the case $\beta_2>0,\beta_3>0$, we have

$$\begin{aligned}
&E\int_0^\tau u_1(s)|y_s|^2\mathrm{d}s+E\int_0^\tau u_2(s)(|q_s|^2+\|p_s\|^2\mathrm{d}s)\\
&\le K_\varepsilon+\varepsilon(\|x\cdot\|_1^2+\|y\cdot\|_1^2+\|q\cdot\|_2^2+\|p\cdot\|_3^2).
\end{aligned}\tag{2.5}$$

Finally, by Itô's formula,

$$\begin{aligned}
E|x_t|^2=&|x_0|^2+E\int_0^t2x_s\cdot b(s,u_s)\mathrm{d}s+E\int_0^t(|\sigma(s,u_s)|^2+\|c(s,u_s,\cdot)\|^2)\mathrm{d}s\\
\le&K+KE\int_0^t(u_1(s)+u_2(s))|x_s|^2\mathrm{d}s+KE\int_0^tu_1(s)|y_s|^2\mathrm{d}s+\varepsilon\|x\cdot\|^2+\\
&KE\int_0^\tau u_2(s)(|q_s|^2+\|p_s\|^2)\mathrm{d}s+\varepsilon^{-1}E(\int_0^\tau|b_2(s,0,0,0,0)|\mathrm{d}s)^2\\
\le&\begin{cases}K_\varepsilon+K(\|y\cdot\|^2+\|q\cdot\|^2+\|p\cdot\|^2)+\varepsilon\|x\cdot\|^2, & \beta_1,\beta_4>0\\ K_\varepsilon+K\varepsilon(\|x\cdot\|^2+\|y\cdot\|^2+\|q\cdot\|^2+\|p\cdot\|^2), & \beta_2,\beta_3>0.\end{cases}
\end{aligned}$$

By the martingale inequality, and taking a small enough $\varepsilon>0$ first, we get

$$\|x\cdot\|^2\le\begin{cases}K+K(\|y\cdot\|^2+\|q\cdot\|^2+\|p\cdot\|^2) & \beta_1,\beta_4>0,\\ K_\varepsilon+K\varepsilon(\|y\cdot\|^2+\|q\cdot\|^2+\|p\cdot\|^2) & \beta_2,\beta_3>0.\end{cases}$$

With the help of this and above inequalities, the assertion follows.

3. Existence and uniqueness

Now we give the result of existence and uniqueness for solutions of FBSDEJs (1.1) and (1.2). Because of the non-Lipschitzian coefficients, we use smoothness techniques in [7]. First the following lemma is necessary, which is the same as that in [12].

Lemma 3.1 *Let the assumptions (H2)–(H5) be satisfied, for $A_1(s,u_s)=0$, then FBSDEJs (1.1) and (1.2) have a unique adapted solution (x_s,y_s,q_s,p_s).*

Remark In [12], we pointed out, under weaker conditions, the uniqueness of solutions for FBSDEJs still holds. In fact, the assumptions (2.1) and (2.2) are equivalent to that in [12].

Theorem 3.2 *Assume that (H1)–(H5) are satisfied. Then there exists a unique solution to the FBSDEJs (1.1) and (1.2).*

Proof We first smooth A by A^n with respect to x and y. Define

$$f^n(s,x,y,q,p)=\int_{R^{n+n}} f(s,x-n^{-1}\bar{x},y-n^{-1}\bar{y},q,p)J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y},$$

where $f=b,\ h,\ \sigma,\ c$ and $J(x,y)=J(x)J(y)$,

$$J(x)=\begin{cases} c_0\ \exp(-(1-|x|^2)^{-1}), & \text{as}\quad |x|<1\\ 0, & \text{otherwise,}\end{cases}$$

and the constant c_0 satisfies $\int_{R^n} J(x)dx=1$. It is easy to check that

$$E\Big(\int_0^T |b^n(s,0,0,0,0)|\mathrm{d}s)^2+\Big(\int_0^T |h^n(s,0,0,0,0)|\mathrm{d}s)^2+$$
$$\int_0^T |\sigma^n(s,0,0,0,0)|^2\mathrm{d}s+\int_0^T \|c^n(s,0,0,0,0,\cdot)\|^2\mathrm{d}s\le K^*<\infty,$$

where K^* is independent of n, and

$$\begin{aligned}
&|f^n(s,x_1,y_1,q_1,p_1)-f^n(s,x_2,y_2,q_2,p_2)|\\
&\quad\le|\int_{R^{n+n}}(f_1(s,x_1-n^{-1}\bar{x},y_1-n^{-1}\bar{y},q_1,p_1)-\\
&\qquad f_1(s,x_2-n^{-1}\bar{x},y_2-n^{-1}\bar{y},q_2,p_2))J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}|+\\
&\quad\int_{R^{n+n}}|f_2(s,x_1-n^{-1}\bar{x},y_1-n^{-1}\bar{y},q_1,p_1)-\\
&\qquad f_2(s,x_2-n^{-1}\bar{x},y_2-n^{-1}\bar{y},q_2,p_2)|J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\\
&\quad\le K_n u_1(s)[|x_1-x_2|+|y_1-y_2|]+2u_2(s)[|q_1-q_2|+\|p_1-p_2\|],
\end{aligned}$$

where K_n is a constant depending on n ,$f = b,\ h,\ \sigma,\ c$. Observe that

$$\begin{aligned}&\langle u_1-u_2, A^n(s,u_1)-A^n(s,u_2)\rangle\\ &=\int_{R^{n+n}}\langle u_1-u_2, A(s,x_1-n^{-1}\bar{x},y_1-n^{-1}\bar{y},q_1,p_1)-\\ &\quad A(s,x_2-n^{-1}\bar{x},y_2-n^{-1}\bar{y},q_2,p_2)\rangle J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}.\end{aligned}$$

Since the assumption (H5) are satisfied, it is easily checked that (H5) also hold for A^n. Consequently, by Lemma 3.1, for each $n = 1,\ 2,\ \cdots$, there exists a unique solution (x_t, y_t, q_t, p_t) to solve the following FBSDEJs:

$$\begin{aligned}x_t^n =&x_0+\int_0^{t\wedge\tau} b^n(s,x_s^n,y_s^n,q_s^n,p_s^n)\mathrm{d}s+\int_0^{t\wedge\tau}\sigma^n(s,x_s^n,y_s^n,q_s^n,p_s^n)\mathrm{d}W_s+\\ &\int_0^{t\wedge\tau}\int_Z c^n(s,x_{s-}^n,y_{s-}^n,q_s^n,p_s^n,z)\tilde{N}_k(\mathrm{d}s,\mathrm{d}z), \qquad (3.1)\\ y_t^n =&\psi(x_\tau^n)+\int_{t\wedge\tau}^{\tau} h^n(s,x_s^n,y_s^n,q_s^n,p_s^n)\mathrm{d}s-\int_{t\wedge\tau}^{\tau} q_s^n\mathrm{d}W_s-\\ &\int_{t\wedge\tau}^{\tau}\int_Z p_s^n(z)\tilde{N}_k(\mathrm{d}z,\mathrm{d}s). \qquad (3.2)\end{aligned}$$

Denote

$$\begin{aligned}&\widehat{X}_t^{n,m}=x_t^n-x_t^m && \widehat{Y}_t^{n,m}=y_t^n-y_t^m \qquad \widehat{Q}_t^{n,m}=q_t^n-q_t^m\\ &\widehat{P}_t^{n,m}=p_t^n-p_t^m && u_t^n=(x_t^n,y_t^n,q_t^n,p_t^n)\\ &\widehat{b}_t^{n,m}=b^n(t,u_t^n)-b^m(t,u_t^m) && \widehat{\sigma}_t^{n,m}=\sigma^n(t,u_t^n)-\sigma^m(t,u_t^m)\\ &\widehat{c}_t^{n,m}=c^n(t,u_t^n,z)-c^m(t,u_t^m,z) && \widehat{\psi}_t^{n,m}=\psi(x_t^n)-\psi(x_t^m)\end{aligned}$$

then

$$\widehat{X}_t^{n,m}=\int_0^t\widehat{b}_s^{n,m}\mathrm{d}s+\int_0^t\widehat{\sigma}_s^{n,m}\mathrm{d}W_s+\int_0^t\int_Z\widehat{c}_s^{n,m}(z)\tilde{N}_k(\mathrm{d}z,\mathrm{d}s), \qquad (3.3)$$

$$\widehat{Y}_t^{n,m}=\widehat{\psi}_\tau^{n,m}+\int_{t\wedge\tau}^{\tau}\widehat{h}_s^{n,m}\mathrm{d}s-\int_{t\wedge\tau}^{\tau}\widehat{Q}_s^{n,m}\mathrm{d}W_s-\int_{t\wedge\tau}^{\tau}\int_Z\widehat{P}_s^{n,m}\tilde{N}_k(\mathrm{d}z,\mathrm{d}s). \qquad (3.4)$$

By Itô's formula and Lemma 2.2(It is true for (3.1) and (3.2), by checking the proof), we have

$$\begin{aligned}&E|\widehat{Y}_t^{n,m}|^2+\frac{1}{2}E\int_{t\wedge\tau}^{\tau}|\widehat{Q}_s^{n,m}|^2\mathrm{d}s+\frac{1}{2}E\int_{t\wedge\tau}^{\tau}\|\widehat{P}_s^{n,m}\|^2\mathrm{d}s\\ &\le E\int_{t\wedge\tau}^{\tau}\int_{R^{n+n}}2u_1(s)\rho(|\widehat{Y}_s^{n,m}-n^{-1}\bar{y}+m^{-1}\bar{y}|^2)J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s+\\ &C^2E|\widehat{X}_\tau^{n,m}|^2+E\int_t^{\tau}2u_1(s)|\widehat{X}_s^{n,m}|^2\mathrm{d}s+\\ &E\int_{t\wedge\tau}^{\tau}(7u_1(s)+8u_2^2(s))|\widehat{Y}_s^{n,m}|^2\mathrm{d}s+|n^{-1}-m^{-1}|^2\widetilde{C}. \qquad (3.5)\end{aligned}$$

Next, use Itô's formula and Lemma 2.2 to get

$$\begin{aligned}
&\beta_4 E|\widehat{X}_\tau^{n,m}|^2 + E\int_0^\tau (\beta_1 u_1(s)|\widehat{X}_s^{n,m}|^2 + \beta_2 u_1(s)|\widehat{Y}_s^{n,m}|^2)\mathrm{d}s + \\
&\quad \beta_3 E\int_0^\tau (|Q_s^{n,m}|^2 + \|P_s^{n,m}\|^2)\mathrm{d}s \\
&\le E\int_0^\tau\int_{R^{n+n}} 2u_1(s)|\widehat{X}_s^{n,m}||n^{-1}\bar{x} - m^{-1}\bar{x}|J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s + \\
&\quad E\int_0^\tau\int_{R^{n+n}} |n^{-1}\bar{x} - m^{-1}\bar{x}||h(s, x_s^m - n^{-1}\bar{x}, y_s^m - n^{-1}\bar{y}, q_s^m - p_s^m) - \\
&\quad h(s, x_s^m - m^{-1}\bar{x}, y_s^m - m^{-1}\bar{y}, q_s^m - p_s^m)|J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s + \\
&\quad E\int_0^\tau\int_{R^{n+n}} 2u_1(s)|\widehat{y}_s^{n,m}||n^{-1}\bar{y} - m^{-1}\bar{y}|J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s + \\
&\quad E\int_0^\tau\int_{R^{n+n}} |n^{-1}\bar{x} - m^{-1}\bar{x}||b(s, x_s^m - n^{-1}\bar{x}, y_s^m - n^{-1}\bar{y}, q_s^m - p_s^m) - \\
&\quad b(s, x_s^m - m^{-1}\bar{x}, y_s^m - m^{-1}\bar{y}, q_s^m - p_s^m)|J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s \\
&\le \widetilde{C}_0|n^{-1} - m^{-1}|,
\end{aligned}$$

and hence

$$\begin{cases} E|\widehat{X}_\tau^{n,m}|^2 + E\int_0^\tau u_1(s)|\widehat{X}_s^{n,m}|^2\mathrm{d}s \longrightarrow 0 & \beta_1,\beta_4 > 0 \\ E\int_0^\tau u_1(s)|\widehat{Y}_s^{n,m}|^2\mathrm{d}s + E\int_0^\tau u_2(s)(|\widehat{Q}_s^{n,m}|^2 + \|\widehat{P}_s^{n,m}\|^2)\mathrm{d}s \longrightarrow 0 & \beta_2,\beta_3 > 0. \end{cases}$$

For the case $\beta_1, \beta_4 > 0$, by (3.5) and Fadou's lemma, we get

$$\overline{\lim_{n,m\to\infty}} E|\widehat{Y}_t^{n,m}|^2 \le \widetilde{K}_0\int_0^\infty (u_1(s) + u_2^2(s))(\rho(\overline{\lim_{n,m\to\infty}} E|\widehat{Y}_s^{n,m}|^2) + \overline{\lim_{n,m\to\infty}} E|\widehat{Y}_s^{n,m}|^2)\mathrm{d}s$$

since $\sup_{0\le t\le\tau} |\widehat{Y}_t^{n,m}|^2 \le K_0$ according to Lemma 2.2. Thus, by SiTu[9] and taking $\rho_1(u) = \rho(u) + u$, it follows that

$$\lim_{m,n\to\infty}(\|\widehat{Y}_t^{n,m}\|_1^2 + \|\widehat{Q}^{n,m}\cdot\|_2^2 + \|\widehat{Q}^{n,m}\cdot\|_3^2) = 0. \tag{3.6}$$

Utilizing the same technique, we also obtain $\lim_{m,n\to\infty}\|\widehat{X}_t^{n,m}\cdot\|_1^2 = 0$.

For the case $\beta_2, \beta_3 > 0$, using Itô's formula to $|\widehat{X}_t^{n,m}|^2$, we have

$$\begin{aligned}
E|\widehat{X}_t^{n,m}|^2 \le & KE\int_0^t\int_{R^{n+n}} 2(u_1(s) + u_2(s))(\rho(|\widehat{X}_s^{n,m} - n^{-1}\bar{x} + m^{-1}\bar{x}|^2) + \\
& |\widehat{X}_s^{n,m} - n^{-1}\bar{x} + m^{-1}\bar{x}|^2)\mathrm{d}s + KE\int_0^t u_1(s)|\widehat{Y}_s^{n,m}|^2\mathrm{d}s + \\
& KE\int_0^t u_2(s)(|\widehat{Q}_s^{n,m}|^2 + \|\widehat{P}_s^{n,m}\|^2)\mathrm{d}s + K|n^{-1} - m^{-1}|^2.
\end{aligned}$$

Hence

$$\overline{\lim_{n,m\to\infty}} E|\widehat{X}_t^{n,m}|^2 \le K\int_0^\infty (u_1(s) + u_2^2(s))(\rho(\overline{\lim_{n,m\to\infty}} E|\widehat{X}_s^{n,m}|^2) + \overline{\lim_{n,m\to\infty}} E|\widehat{X}_s^{n,m}|^2)\mathrm{d}s.$$

This gives $\lim_{n,m\to\infty} E|\widehat{X}_t^{n,m}|^2 = 0$. From this and the martigale inequality, it deduces that

$$\lim_{n,m\to\infty}\|\widehat{X}^{n,m}\cdot\|_1^2 = 0.$$

Similarly, we also have

$$\lim_{n,m\to\infty}(\|\widehat{X}^{n,m}\cdot\|_1^2+\|Q^{n,m}\cdot\|_2^2+\|P^{n,m}\cdot\|_3^2)=0.$$

Therefore, there exists a unique $u_t=(x_t,y_t,q_t,p_t)\in S^2_{\mathcal{F}_t}(R^n)\times S^2_{\mathcal{F}_t}(R^n)\times L^2_{\mathcal{F}_t}(R^{n\times d})\times F^2_{\mathcal{F}_t}(R^n)$. We can take a subsequence $\{n_k\}$ of $\{n\}$, and denote it by $\{n\}$ again, such that as $n\to\infty$

$$\begin{aligned}
&x_t^n(\omega)\to x_t(\omega) \quad \text{in} \quad R^n\\
&y_t^n(\omega)\to y_t(\omega) \quad \text{in} \quad R^n\\
&q_t^n(\omega)\to q_t(\omega) \quad \text{in} \quad R^{n\times d}\\
&p_t^n(\omega)\to y_t^n(\omega) \quad \text{in} \quad L^2_{\Pi(\cdot)}(R^n)
\end{aligned}$$

a.e. $(t,\omega)\in[0,\tau]\times\Omega$. It is easy to derive that as $n\to\infty$

$$E|x_t^n(\omega)-x_t(\omega)|\to 0, \qquad E|y_t^n(\omega)-y_t(\omega)|\to 0.$$

Let us show that as $n\to\infty$

$$E|\int_0^t(b^n(s,u_s^n,\omega)-b(s,u_s,\omega))\mathrm{d}s|\to 0.$$

Note that

$$\begin{aligned}
&E\int_0^\tau\int_{R^{n+n}}|b(s,x_s^n-n^{-1}\bar{x},y_s^n-n^{-1}\bar{y},q_s^n,p_s^n)-b(s,x_s,y_s,q_s,p_s)|^2J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s\\
&\le KE\int_0^\tau\int_{R^{n+n}}\Big(u_1^2(s)(|x_s^n|^2+|y_s^n|^2)+u_1^2(s)(|x_s|^2+|y_s|^2)+\\
&u_2^2(s)(|q_s^n|^2+|q_s|^2)+u_2^2(s)(|q_s|^2+\|p_s\|^2)+\\
&u_1^2(s)+u_1^2(s)(|\bar{x}|^2+|\bar{y}|^2)n^{-2}\Big)J(\bar{x},\bar{y})\mathrm{d}\bar{x}\mathrm{d}\bar{y}\mathrm{d}s,
\end{aligned}$$

since

$$\begin{aligned}
&E[\sup_{0\le t\le\tau}|x_t|^2+\sup_{0\le t\le\tau}|y_t|^2+\int_0^\tau(|q_t|^2+\|p_t\|^2)\mathrm{d}t]\\
&=E[\sup_{0\le t\le\tau}\varliminf_{n\to\infty}|x_t^n|^2+\sup_{0\le t\le\tau}\varliminf_{n\to\infty}|y_t^n|^2+\int_0^\tau\varliminf_{n\to\infty}(|q_t^n|^2+\|p_t^n\|^2)\mathrm{d}t]\\
&\le\varliminf_{n\to\infty}E[\sup_{0\le t\le\tau}|x_t^n|^2+\sup_{0\le t\le\tau}|y_t^n|^2+\int_0^\tau(|q_t^n|^2+\|p_t^n\|^2)\mathrm{d}t]\le K_0.
\end{aligned}$$

Hence the sequence $\{|b(s,x_s^n-n^{-1}\bar{x},y_s^n-n^{-1}\bar{y},q_s^n,p_s^n)-b(s,x_s,y_s,q_s,p_s)|^2\}_{n=1}^\infty$ is uniformly integrable on $[0,\tau]\times R^n\times R^n\times\Omega$. Recalling that b is continuous, we immediately deduce that as $n\to\infty$

$$E|\int_0^t(b^n(s,u_s^n,\omega)-b(s,u_s,\omega))\mathrm{d}s|\to 0.$$

Analogously, it is shown that

$$E|\int_0^t (h^n(s,u_s^n,\omega)-h(s,u_s,\omega))\mathrm{d}s|\to 0$$

Now we show that

$$E|\int_0^t (\sigma^n(s,x_s^n,y_s^n,q_s^n,p_s^n)-\sigma(s,x_s,y_s,q_s,p_s))\mathrm{d}W_s|\to 0.$$

Indeed, by Lemma 2.2, for each n we get

$$E|\int_0^t (\sigma^n(s,x_s^n,y_s^n,q_s^n,p_s^n)-\sigma(s,x_s,y_s,q_s,p_s))\mathrm{d}W_s|^2\le K_0.$$

It means that $\{|\int_0^t(\sigma^n(s,x_s^n,y_s^n,q_s^n,p_s^n)-\sigma(s,x_s,y_s,q_s,p_s))\mathrm{d}W_s|\}_{n=1}^{\infty}$ is uniformly integrable. Observe that for each $\varepsilon>0$, by Doob's inequality, it deduces that

$$P(|\int_0^t (\sigma^n(s,u_s^n)-\sigma(s,u_s))\mathrm{d}W_s|\ge\varepsilon)\le\frac{1}{\varepsilon^2}E\int_0^T|\sigma^n(s,u_s^n)-\sigma(s,u_s)|^2\mathrm{d}s.$$

Therefore, by Lebesgue's domination convergence theorem and the continuity of σ, we have

$$P(|\int_0^t (\sigma^n(s,u_s^n)-\sigma(s,u_s))\mathrm{d}W_s|\ge\varepsilon)\longrightarrow 0,$$

which implies that

$$E|\int_0^t (\sigma^n(s,x_s^n,y_s^n,q_s^n,p_s^n)-\sigma(s,x_s,y_s,q_s,p_s))\mathrm{d}W_s|\to 0.$$

Similarly,

$$E|\int_0^t (c^n(s,x_s^n,y_s^n,q_s^n,p_s^n,z)-c(s,x_s,y_s,q_s,p_s,z))\widetilde{N}_k(\mathrm{d}z,\mathrm{d}s)|\to 0$$

is also hold. Now let $n\to\infty$ in (3.1) and (3.2), we obtain that (x_t,y_t,q_t,p_t) is a solution of (1.1) and (1.2). Observe that, from Schwarz's inequality, if $E\int_0^T u_1(s)|x_s|^2\mathrm{d}s=0$, then $E\int_0^T u_1(s)|x_s|\mathrm{d}s=0$, the proof of uniqueness is similar to that of Theorem 2.1 in [12]. Thus, the assertion is followed.

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具有跳跃的非 Lipschitz 系数正 - 倒向随机微分方程解的存在性

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摘 要: 研究了终端为停时带 Poisson 跳的正 - 倒向随机微分方程. 在非 Lipschitz 系数和弱单调性的假设条件下, 应用概率分析方法, 证明了方程解的存在唯一性, 同时给出了有关的先验估计. 其中的正向方程允许为退化情形.

关键词: 正 - 倒向随机微分方程; 无界停时; 非 Lipschitz 系数; 先验估计.