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# On Monogeny and Epigeny Classes of Modules *

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**Abstract**: Let $(S,\leq)$ be a strictly totally ordered monoid, and $M$ and $N$ be left $R$-modules. We show the following results: (1) If $(S,\leq)$ is finitely generated and satisfies the condition that $0\leq s$ for any $s\in S$, then $\mathrm{Epi}_{([[R^{S,\leq}]]}[[M^{S,\leq}]])=\mathrm{Epi}_{([[R^{S,\leq}]]}[[N^{S,\leq}]])$ if and only if $\mathrm{Epi}(M)=\mathrm{Epi}(N)$; (2) If $(S,\leq)$ is artinian, then $\mathrm{Mono}_{([[R^{S,\leq}]]}[M^{S,\leq}])=\mathrm{Mono}_{([[R^{S,\leq}]]}[N^{S,\leq}])$ if and only if $\mathrm{Mono}(M)=\mathrm{Mono}(N)$.

**Key words**: Monogeny class; Epigeny class; generalized power series ring .

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## 1. Prelimilaries

All rings considered here are associative with identity and all modules are unital unless otherwise specified. Any concept and notation not defined here can be found in [9].

Following [1] or [2], two left $R$-modules $M$ and $N$ are said to belong to the same monogeny class, if there are a monomophism $f:M\longrightarrow N$ and a monomorphism $g:N\longrightarrow M$. In this case we denote $\mathrm{Mono}(M)=\mathrm{Mono}(N)$. Similarly, $M$ and $N$ are in the same epigeny class if there are an epimorphism $h:M\longrightarrow N$ and an epimorphism $k:N\longrightarrow M$. In this case we denote $\mathrm{Epi}(M)=\mathrm{Epi}(N)$. In [1] and [2], the authors discussed finite direct sums of uniserial modules and the weak form of the Krull-Schmidt Theorem for serial modules, respectively, by using the concepts of monogeny class and epigeny class of modules. In this paper we consider the influence of extensions of generalized power series rings upon the properties of belonging to the same monogeny class and epigeny class of modules.

Let $(S,\leq)$ be an ordered set. Recall that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of $S$ is finite, and that $(S,\leq)$ is narrow if every subset of pairwise order-incomparable elements of $S$ is finite. Let $S$ be a commutative monoid. Unless stated

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otherwise, the operation of $S$ shall be denoted additively, and the neutral element by 0. The following definition is due to [9].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s+t < s'+t$), and $R$ a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f: S \longrightarrow R$ such that $\mathrm{supp}(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let $X(s; f, g) = \{(u, v) \in S \times S | s = u + v, f(u) \neq 0, g(v) \neq 0\}$. It follows from [9, 4.1] that $X(s; f, g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X(s;f,g)} f(u)g(v).$$

Clearly, $\mathrm{supp}(fg) \subseteq \mathrm{supp}(f) + \mathrm{supp}(g)$, thus by [9, 2.1], $\mathrm{supp}(fg)$ is artinian and narrow, hence $fg \in [[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in $R$ and exponents in $S$.

For example, if $S = \mathbf{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbf{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If $S$ is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] = R[S]$, the monoid-ring of $S$ over $R$. Further examples are given in [5, 9–12].

## 2. Epigeny classes of modules of generalized power series

Let $M$ be a left $R$-module. We denote by $[[M^{S,\leq}]]$ the set of all maps $\varphi: S \longrightarrow M$ such that $\mathrm{supp}(\varphi) = \{s \in S | \varphi(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[M^{S,\leq}]]$ is an abelian additive group. For each $f \in [[R^{S,\leq}]]$, each $\varphi \in [[M^{S,\leq}]]$ and $s \in S$, denote

$$X(s; f, \varphi) = \{(u, v) \in S \times S | s = u + v, f(u) \neq 0, \varphi(v) \neq 0\}.$$

By [4], $X(s; f, \varphi)$ is finite. Thus from [4], $[[M^{S,\leq}]]$ can be turned into a left $[[R^{S,\leq}]]$-module by the scalar multiplication defined as follows

$$(f\varphi)(s) = \sum_{(u,v)\in X(s;f,\varphi)} f(u)\varphi(v)$$

for each $f \in [[R^{S,\leq}]]$ and each $\varphi \in [[M^{S,\leq}]]$. $[[M^{S,\leq}]]$ is called the module of generalized power series over left $R$-module $M$. The elements of $[[M^{S,\leq}]]$ are called generalized power series with coefficients in $M$ and exponents in $S$.

For every $\varphi \in [[M^{S,\leq}]]$ with $\varphi \neq 0$, $\mathrm{supp}(\varphi)$ is a nonempty well-ordered subset of $S$. We denote by $\pi(\varphi)$ the smallest element of the support of $\varphi \neq 0$. As in [10] we adjoin an element $\infty$ to $S$ with $s < \infty$ and $s + \infty = \infty + s = \infty$ for all $s \in S$ and define $\pi(0) = \infty$.

If $s_1, \ldots, s_n \in S$ we denote by $< s_1, \ldots, s_n >$ the set of all elements $\sum_{i=1}^{n} k_i s_i$ (with $k_i$ integer, $k_i \geq 0$). A monoid $S$ is called finitely generated if there exists a finite subset $\{s_1, \ldots, s_n\}$ such that $S =< s_1, \ldots, s_n >$.

For any $s \in S$, define $e_s \in [[R^{S,\leq}]]$ as follows:

$$e_s(x) = \begin{cases} 1 & x = s \\ 0 & x \neq s. \end{cases}$$

By analogy with the proof of [4, Lemma 3], we have

**Lemma 2.1** *Suppose that $(S,\leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0\leq s$ for any $s\in S$. Assume that $\alpha\in \mathrm{Hom}_{[[R^{S,\leq}]]}([[M^{S,\leq}]],[[N^{S,\leq}]])$ and $0\neq\varphi\in[[M^{S,\leq}]]$. Then $\pi(\alpha(\varphi))\geq\pi(\varphi)$.*

**Lemma 2.2** *Let $M$ and $N$ be left $R$-modules. If $\mathrm{Epi}(M)=\mathrm{Epi}(N)$, then*

$$\mathrm{Epi}(_{[[R^{S,\leq}]]}[[M^{S,\leq}]])=\mathrm{Epi}(_{[[R^{S,\leq}]]}[[N^{S,\leq}]]).$$

**Proof** Let $h: M\to N$ be a surjective $R$-homomorphism. Define $\alpha:[[M^{S,\leq}]]\to[[N^{S,\leq}]]$ via

$$\begin{aligned}\alpha(\varphi):\ & S\to M\\ & s\to h(\varphi(s))\end{aligned}$$

for any $\varphi\in[[M^{S,\leq}]]$.

For any $f\in[[R^{S,\leq}]]$, any $\varphi\in[[M^{S,\leq}]]$ and any $s\in S$, set $X_1=\{(u,v)\in X(s;f,\varphi)|h(\varphi(v))=0\}$, $X_2=\{(u,v)\in X(s;f,\varphi)|h(\varphi(v))\neq 0\}$. Then clearly $X_2=X(s;f,\alpha(\varphi))$. Now

$$\begin{aligned}\alpha(f\varphi)(s)=h((f\varphi)(s))&=h(\sum_{(u,v)\in X(s;f,\varphi)}f(u)\varphi(v))\\ &=\sum_{(u,v)\in X(s;f,\varphi)}f(u)h(\varphi(v))\\ &=\sum_{(u,v)\in X_1}f(u)h(\varphi(v))+\sum_{(u,v)\in X_2}f(u)h(\varphi(v))\\ &=\sum_{(u,v)\in X(s;f,\alpha(\varphi))}f(u)\alpha(\varphi)(v)\\ &=(f\alpha(\varphi))(s).\end{aligned}$$

This means that $\alpha(f\varphi)=f\alpha(\varphi)$. Thus $\alpha$ is an $[[R^{S,\leq}]]$-homomorphism.

For any $\psi\in[[N^{S,\leq}]]$ and any $s\in\mathrm{supp}(\psi)$, there exists an element $m_s\in M$ such that $h(m_s)=\psi(s)$ since $h$ is surjective. Define $\varphi:S\to M$ via

$$\varphi(s)=\begin{cases} m_s & s\in\mathrm{supp}(\psi)\\ 0 & s\notin\mathrm{supp}(\psi).\end{cases}$$

Clearly $\mathrm{supp}(\varphi)\subseteq\mathrm{supp}(\psi)$, which implies that $\mathrm{supp}(\varphi)$ is artinian and narrow and thus $\varphi\in[[M^{S,\leq}]]$. If $s\in\mathrm{supp}(\psi)$, then $\alpha(\varphi)(s)=h(\varphi(s))=h(m_s)=\psi(s)$. If $s\notin\mathrm{supp}(\psi)$, then $\alpha(\varphi)(s)=h(\varphi(s))=0$. Thus $\alpha(\varphi)=\psi$. This means that $\alpha$ is a surjective $[[R^{S,\leq}]]$-homomorphism. Hence there exists an $[[R^{S,\leq}]]$-epimorphism $\alpha:[[M^{S,\leq}]]\to[[N^{S,\leq}]]$.

Similarly there exists an $[[R^{S,\leq}]]$-epimorphism $\beta:[[N^{S,\leq}]]\to[[M^{S,\leq}]]$. Thus

$$\mathrm{Epi}(_{[[R^{S,\leq}]]}[[M^{S,\leq}]])=\mathrm{Epi}(_{[[R^{S,\leq}]]}[[N^{S,\leq}]]).$$

**Lemma 2.3** *Suppose that $(S,\leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0\leq s$ for any $s\in S$. If $M$ and $N$ be left $R$-modules such that $\mathrm{Epi}(_{[[R^{S,\leq}]]}[[M^{S,\leq}]])=\mathrm{Epi}(_{[[R^{S,\leq}]]}[[N^{S,\leq}]])$, then $\mathrm{Epi}(_RM)=\mathrm{Epi}(_RN)$.*

**Proof** Let $\alpha : [[M^{S,\leq}]] \to [[N^{S,\leq}]]$ be a surjective $[[R^{S,\leq}]]$-homomorphism. Define $f : [[N^{S,\leq}]] \to N$ via $f(\varphi) = \varphi(0)$. For any $m \in M$, define $\lambda_m \in [[M^{S,\leq}]]$ as

$$\lambda_m(x) = \begin{cases} m & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Now define $h : M \longrightarrow N$ via $h(m) = f\alpha(\lambda_m)$. For any $r \in R$, define $c_r \in [[R^{S,\leq}]]$ via $c_r(0) = r$ and $c_r(x) = 0$ for all $0 \neq x \in S$. Then $h(rm) = f\alpha(\lambda_{rm}) = f\alpha(c_r\lambda_m) = f(c_r\alpha(\lambda_m)) = (c_r\alpha(\lambda_m))(0) = r\alpha(\lambda_m)(0) = rf\alpha(\lambda_m) = rh(m)$. Thus $h$ is an $R$-homomorphism. For any $n \in N$, there exists $\varphi \in [[M^{S,\leq}]]$ such that $\alpha(\varphi) = \lambda_n$. Let $\psi = \varphi - \lambda_{\varphi(0)}$. Then $\pi(\psi) > 0$ and thus $\pi(\alpha(\psi)) > 0$ by Lemma 2.1. Hence

$$\begin{aligned} n = \lambda_n(0) &= \alpha(\varphi)(0) = (\alpha(\psi + \lambda_{\varphi(0)}))(0) \\ &= (\alpha(\psi))(0) + (\alpha(\lambda_{\varphi(0)}))(0) = f\alpha(\lambda_{\varphi(0)}) = h(\varphi(0)). \end{aligned}$$

This means that $h$ is a surjective $R$-homomorphism. Hence there exists an $R$-epimorphism $h : M \to N$.

Similarly there exists an $R$-epimorphism $k : N \to M$. Thus $\mathrm{Epi}({}_RM) = \mathrm{Epi}({}_RN)$.

**Theorem 2.4** *Suppose that $(S, \leq)$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that $0 \leq s$ for any $s \in S$. Then for left $R$-modules $M$ and $N$, the following conditions are equivalent:*

*(1)* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$.

*(2)* $\mathrm{Epi}_{([[R^{S,\leq}]]}[[M^{S,\leq}]]) = \mathrm{Epi}_{([[R^{S,\leq}]]}[[N^{S,\leq}]])$.

**Proof** It follows from Lemmas 2.2 and 2.3.

## 3. Monogeny classes of modules of generalized inverse polynomials

If $M$ is a left $R$-module, we let $[M^{S,\leq}]$ be the set of all maps $\varphi : S \to M$ such that the set $\mathrm{supp}(\varphi) = \{s \in S | \varphi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows

$$(f\varphi)(s) = \sum_{t \in S} f(t)\varphi(s + t), \quad \text{for every} \quad s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\varphi \in [M^{S,\leq}]$. By [6], if $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then $\mathrm{supp}(f\varphi)$ is finite and $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module. The elements of $[M^{S,\leq}]$ are called generalized inverse polynomials with coefficients in $M$ and exponents in $S$.

Note that the usual left $R[[x]]$-module $M[x^{-1}]$ introduced in [8] is a module of generalized inverse polynomials. Further examples of modules of generalized inverse polynomials are given in [5, 6].

Suppose that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian. If $s \in S$ is such that $s < 0$, then

$$\cdots < 3s < 2s < s < 0$$

by [10, 3.1]. This contradicts the assumption that $(S, \leq)$ is artinian. Thus for any $s \in S$, we have $0 \leq s$. This result will be often used throughout this section.

For every $\varphi \in [M^{S,\leq}]$, we denote by $\sigma(\varphi)$ the maximal element in $\mathrm{supp}(\varphi)$. The following result is a direct corollary of [7, Lemma 2.1].

**Lemma 3.1** *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian. Suppose $\alpha \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}])$. Then for any $\varphi \in [M^{S,\leq}]$, $\sigma(\alpha(\varphi)) \leq \sigma(\varphi)$.*

**Theorem 3.2** *Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian. Let $M$ and $N$ be left $R$-modules. Then $\mathrm{Mono}({}_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}({}_{[[R^{S,\leq}]]}[N^{S,\leq}])$ if and only if $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.*

**Proof** Suppose $\mathrm{Mono}({}_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}({}_{[[R^{S,\leq}]]}[N^{S,\leq}])$. Let

$$\alpha \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}])$$

be injective. For any $a \in M$, define $\varphi_{0a} \in [M^{S,\leq}]$ as follows

$$\varphi_{0a}(t) = \begin{cases} a, & t = 0 \\ 0, & t \neq 0 \end{cases} \quad t \in S.$$

Since $\sigma(\varphi_{0a}) = 0$, it follows from Lemma 3.1 that $\sigma(\alpha(\varphi_{0a})) \leq 0$. Thus $\alpha(\varphi_{0a})(s) = 0$ for any $0 \neq s \in S$.

Define $f : M \to N$ via

$$f(a) = \alpha(\varphi_{0a})(0), \quad \forall a \in M.$$

For any $r \in R$,

$$(c_r\varphi_{0a})(x) = \sum_{y \in S} c_r(y)\varphi_{0a}(x + y) = r\varphi_{0a}(x) = \varphi_{0,ra}(x).$$

Thus, $c_r\varphi_{0a} = \varphi_{0,ra}$, and so

$$\begin{aligned} f(ra) &= \alpha(\varphi_{0,ra})(0) = \alpha(c_r\varphi_{0a})(0) = c_r\alpha(\varphi_{0a})(0) \\ &= \sum_{y \in S} c_r(y)\alpha(\varphi_{0a})(y) = r\alpha(\varphi_{0a})(0) = rf(a). \end{aligned}$$

This means that $f$ is an $R$-homomorphism. If $f(a) = 0$, then clearly $\alpha(\varphi_{0a}) = 0$, which implies that $\varphi_{0a} = 0$ since $\alpha$ is injective. Thus $a = 0$. This means that $f$ is an injective $R$-homomorphism. Hence there exists an $R$-monomorphism $h : M \to N$.

Similarly, there exists an $R$-monomorphism $k : N \to M$. Thus $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.

Conversely, suppose $\mathrm{Mono}(M) = \mathrm{Mono}(N)$. Let $f \in Hom_R(M, N)$ be injective. Define $\alpha : [M^{S,\leq}] \to [N^{S,\leq}]$ via

$$\alpha(\varphi)(s) = f(\varphi(s)), \quad s \in S, \quad \varphi \in [M^{S,\leq}].$$

Clearly, for any $g \in [[R^{S,\leq}]]$ and any $s \in S$,

$$\begin{aligned}\alpha(g\varphi)(s) &= f[(g\varphi)(s)] = f(\sum_{y\in S} g(y)\varphi(s+y)) \\ &= \sum_{y\in S} g(y) f(\varphi(s+y)) \\ &= \sum_{y\in S} g(y)[\alpha(\varphi)](s+y) \\ &= (g\alpha(\varphi))(s),\end{aligned}$$

which implies that $\alpha$ is an $[[R^{S,\leq}]]$-homomorphism. If $\alpha(\varphi) = 0$, then for any $s \in S$, $f(\varphi(s)) = 0$, which implies that $\varphi(s) = 0$ since $f$ is injective. Thus $\varphi = 0$. This means that $\alpha$ is injective. Hence there exists an $[[R^{S,\leq}]]$-monomorphism $\alpha : [M^{S,\leq}] \to [N^{S,\leq}]$.

Similarly, there exists an $[[R^{S,\leq}]]$-monomorphism $\beta : [N^{S,\leq}] \to [M^{S,\leq}]$. Thus

$$\mathrm{Mono}({}_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}({}_{[[R^{S,\leq}]]}[N^{S,\leq}]).$$

**Remark 3.3** Let $R$ be a ring not necessarily possessing an identity. Varadarajan[14,15] says that a left $R$-module $M$ has property (F) if for any submodule $N$ of $M$ we have $\{m \in M | Rm \leq N\} = N$. It is easy to see that $M$ has property (F) if and only if $m \in Rm$ for every $m \in M$, i.e., $M$ is an $s$-unital module in the sense of Tominaga[13]. It follows from [13, Theorem 1] that $M$ has property (F) if and only if for any finitely many elements $m_1, \ldots, m_n \in M$ there exists an element $e \in R$ such that $em_i = m_i$, $i = 1, \ldots, n$. Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian. It was proved in [7] that if $N$ has property (F) and $\alpha \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}])$ then for any $\varphi \in [M^{S,\leq}]$, $\sigma(\alpha(\varphi)) \leq \sigma(\varphi)$. Using this result, by analogy with the proof of Theorem 3.2, we can show that if $M$ and $N$ are left $R$-modules having property (F), then $\mathrm{Mono}({}_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}({}_{[[R^{S,\leq}]]}[N^{S,\leq}])$ if and only if $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.

## 4. Corollaries

**Corollary 4.1** *Let $S$ be a finitely generated torsion-free and cancellative monoid, and $(S, \leq)$ be narrow and satisfy the condition that $0 \leq s$ for every $s \in S$. Then for left $R$-modules $M$ and $N$, we have* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$ *if and only if* $\mathrm{Epi}({}_{[[R^{S,\leq}]]}[[M^{S,\leq}]]) = \mathrm{Epi}({}_{[[R^{S,\leq}]]}[[N^{S,\leq}]])$.

**Proof** If $(S, \leq)$ is torsion-free and cancellative, then by [10, 3.3], there exists a compatible strict total order $\leq'$ on $S$, which is finer than $\leq$, that is, for any $s, t \in S$, $s \leq t$ implies $s \leq' t$. By [12], we have $0 \leq' s$ for any $s \in S$. Thus by Theorem 2.4, $\mathrm{Epi}(M) = \mathrm{Epi}(N)$ if and only if $\mathrm{Epi}({}_{[[R^{S,\leq'}]]}[[M^{S,\leq'}]]) = \mathrm{Epi}({}_{[[R^{S,\leq'}]]}[[N^{S,\leq'}]])$. On the other hand, since $(S, \leq)$ is narrow, by [10, 4.4], $[[R^{S,\leq}]] = [[R^{S,\leq'}]]$. By analogy with the proof of [10, 4.4], it follows that $[[M^{S,\leq}]] = [[M^{S,\leq'}]]$. Now the result follows.

Any submonoid of the additive monoid $\mathbf{N} \cup \{0\}$ is called a numerical monoid. It is well-known that any numerical monoid is finitely generated (see [10, 1.3] or [3, p. 13]). Thus we have

**Corollary 4.2** *Let $S$ be a numerical monoid and $\leq$ the usual natural order of $\mathbf{N} \cup \{0\}$. Then for any left $R$-modules $M$ and $N$, we have*

*(1)* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$ *if and only if*

$$\mathrm{Epi}(_{[[R^{S,\leq}]]}[[M^{S,\leq}]]) = \mathrm{Epi}(_{[[R^{S,\leq}]]}[[N^{S,\leq}]]).$$

*(2)* $\mathrm{Mono}(M) = \mathrm{Mono}(N)$ *if and only if*

$$\mathrm{Mono}(_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}(_{[[R^{S,\leq}]]}[N^{S,\leq}]).$$

**Corollary 4.3** *Let $(S_1, \leq_1), \cdots, (S_n, \leq_n)$ be strictly totally ordered monoids. Denote by (lex $\leq$) and (revlex $\leq$) the lexicographic order and the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. Let $M$ and $N$ be left $R$-module.*

*(1) If $(S_i, \leq_i)$ is finitely generated and satisfies the condition that $0 \leq_i s$ for every $s \in S_i, i = 1, \ldots, n$, then the following conditions are equivalent:*

*(i)* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$;

*(ii)* $\mathrm{Epi}([[M^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]]) = \mathrm{Epi}([[N^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]])$;

*(iii)* $\mathrm{Epi}([[M^{S_1\times\cdots\times S_n,(\mathrm{revlex}\leq)}]]) = \mathrm{Epi}([[N^{S_1\times\cdots\times S_n,(\mathrm{revlex}\leq)}]])$.

*(2) If $(S_i, \leq_i)$ is artinian, $i = 1, \cdots, n$, then the following conditions are equivalent:*

*(i)* $\mathrm{Mono}(M) = \mathrm{Mono}(N)$;

*(ii)* $\mathrm{Mono}([M^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}]) = \mathrm{Mono}([N^{S_1\times\cdots\times S_n,(\mathrm{lex}\leq)}])$.

*(iii)* $\mathrm{Mono}([M^{S_1\times\cdots\times S_n,(\mathrm{revlex}\leq)}]) = \mathrm{Mono}([N^{S_1\times\cdot\times S_n,(\mathrm{revlex}\leq)}])$.

**Proof** (1) It is easy to see that $(S_1 \times \cdots \times S_n, (\mathrm{lex} \leq))$ is a strictly totally ordered monoid which is finitely generated and satisfies the condition that

$$(0, \cdots, 0)(\mathrm{lex} \leq)(s_1, \cdots, s_n)$$

for every $(s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n$. Thus, by Theorem 2.4, we have (i)$\Leftrightarrow$(ii). The proof of (i)$\Leftrightarrow$ (iii) is similar.

(2) It follows from Theorem 3.2.

Let $p_1, \cdots, p_n$ be prime numbers. Set

$$N(p_1, \cdots, p_n) = \{p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n} | m_1, m_2, \cdots, m_n \in \mathbf{N} \cup \{0\}\}.$$

Then $N(p_1, \cdots, p_n)$ is a submonoid of $(\mathbf{N}, \cdot)$. Let $\leq$ be the usual natural order.

**Corollary 4.4** *Let $M$ and $N$ be left $R$-modules. Set $A = [[R^{N(p_1,\cdots,p_n),\leq}]]$. Then*

*(1)* $\mathrm{Epi}(_A[[M^{N(p_1,\cdots,p_n),\leq}]]) = \mathrm{Epi}(_A[[N^{N(p_1,\cdots,p_n),\leq}]])$ *if and only if* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$.

*(2)* $\mathrm{Mono}(_A[M^{N(p_1,\cdots,p_n),\leq}]) = \mathrm{Mono}(_A[N^{N(p_1,\cdots,p_n),\leq}])$ *if and only if* $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.

**Corollary 4.5** *Let $x_1, \cdots, x_n$ be $n$ commuting indeterminates over $R$. Let $M$ and $N$ be left $R$-modules. Set $A = R[[x_1, \cdots, x_n]]$.*

*(1)* $\mathrm{Epi}(_AM[[x_1, \cdots, x_n]]) = \mathrm{Epi}(_AN[[x_1, \cdots, x_n]])$ *if and only if* $\mathrm{Epi}(M) = \mathrm{Epi}(N)$.

*(2)* $\mathrm{Mono}(_AM[x_1^{-1}, \cdots, x_n^{-1}]) = \mathrm{Mono}(_AN[x_1^{-1}, \cdots, x_n^{-1}])$ *if and only if* $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.

**Proof** Let $S_1 = \cdots = S_n = \mathbf{N}$ and $\leq_i$, $i = 1, \cdots, n$, be the usual order of $\mathbf{N}$ in Corollary 4.3. Then the result follows from [10, Example 3].

## References:


[1] DUNG N V, FACCHINI A. *Direct summands of serial modules* [J]. J. Pure Appl. Algabra, 1998, **133**(1-2): 93–106.

[2] FACCHINI A. *Krull-Schmidt fails for serial modules* [J]. Trans. Amer. Math. Soc., 1996, **384**(11): 4561–4575.

[3] GILMER R. *Commutative Semigroup Rings* [M]. University of Chicago Press, Chicago, 1984.

[4] LIU Zhong-kui. A note on Hopfian modules [J]. Comm. Algebra, 2000, **28**: 3031–3040.

[5] LIU Zhong-kui. *Endomorphism rings of modules of generalized inverse polynomials* [J]. Comm. Algebra, 2000, **28**: 803–814.

[6] LIU Zhong-kui, CHENG Hui. *Quasi-duality for the rings of generalized power series* [J]. Comm. Algebra, 2000, **28**: 1175–1188.

[7] LIU Zhong-kui, FAN Yuan. *Co-Hopfian modules of generalized inverse polynomials* [J]. Acta Mathematica Sinica, English Series, 2001, **17**: 431–436.

[8] PARK S. *Inverse polynomials and injective covers* [J]. Comm. Algebra, 1993, **21**: 4599–4613.

[9] RIBENBOIM P. *Semisimple rings and von Neumann regular rings of generalized power series* [J]. J. Algebra, 1997, **198**(2): 327–338.

[10] RIBENBOIM P. *Noetherian rings of generalized power series* [J]. J. Pure Appl. Algebra, 1992, **79**(3): 293–312.

[11] RIBENBOIM P. *Rings of generalized power series II: units and zero-divisors* [J]. J. Algebra, 1994, **168**(1): 71–89.

[12] RIBENBOIM P. *Special properties of generalized power series* [J]. J. Algebra, 1995, **173**(3): 566–586.

[13] TOMINAGA H. *On s-unital rings* [J]. Math. J. Okayama Univ., 1976, **18**: 117–134.

[14] VARADARAJAN K. *A generalization of Hilbert's basis theorem* [J]. Comm. Algebra, 1982, **10**: 2191–2204.

[15] VARADARAJAN K. *A note on the Hopficity of $M[X]$ or $M[[X]]$* [J]. Nat. Acad. Sci. Letters, 1992, **15**: 53–56.


# 关于 Monogeny 和 Epigeny 模类


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**摘 要**: 设 $(S, \leq)$ 是严格全序幺半群, $M$ 和 $N$ 是左 $R$- 模. 记 $A = [[R^{S,\leq}]]$. 证明了如下结论: (1) 如果 $(S, \leq)$ 是有限生成的且对任意 $s \in S$ 有 $0 \leq s$, 则 $\mathrm{Epi}(_{[[R^{S,\leq}]]}[[M^{S,\leq}]]) = \mathrm{Epi}(_{[[R^{S,\leq}]]}[[N^{S,\leq}]])$ 当且仅当 $\mathrm{Epi}(M) = \mathrm{Epi}(N)$; (2) 如果 $(S, \leq)$ 是 Artinian 的, 则

$$\mathrm{Mono}(_{[[R^{S,\leq}]]}[M^{S,\leq}]) = \mathrm{Mono}(_{[[R^{S,\leq}]]}[N^{S,\leq}])$$

当且仅当 $\mathrm{Mono}(M) = \mathrm{Mono}(N)$.



**关键词**: Monogeny 类; Epigeny 类; 广义幂级数环.