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Hereditary Radicals and Strongly Semisimple Radicals in Normal Classes of Algebras *

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Abstract: Puczylowski[1] established the general theory of radicals of the class of objects called algebras. In this paper, we make use of the method of lattice theory to characterize the general hereditary radicals and general strongly semisimple radicals and investigate some properties of them in normal classes of algebras. This extends some known studies on the theory of radicals of various algebraic strutures.

Key words: algebra; normal class; lattice; hereditary radical; strongly semisimple radical.

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1. Introduction

Recently, Puczylowski[1] developed some ideas of the theory of radicals of various algebraic structures, established the general theory of radical in the Universal classes of objects called algebras by means of axioms system, gave the characterizations of general radical class and general Semisimple class, and illustrated the possibilities of applications of this theory to three samples. Here, we will propose the concepts of hereditary radical and strongly semisimple radical and present characterizations for them by means of the lattice theory method. As the normal classes of algebras are much more extensive than the class of associative rings, the results of this paper generalize respectively the problems 12-14 and Theorems 8.1-8.2 of [2] for associative rings to any normal class of algebras, such as the category $\mathcal{F}$ of alternative algebras over a commutative ring F with usual identification, the Category φ of Semigroup with zero, the Category g of associative algebra with involution over a Commutative ring F, the Category β of G-graded rings and (h,k)-graded homomorphisms and the category b of Γ-rings, etc., and these results also can be applied to many other categories satisfying axiom A1-A6. In this way one obtain known and new results on radicals of these classes.

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2. Preliminaries

Let $\mathcal{A}$ be a class of objects called algebras, O be a fixed element of $\mathcal{A}$ called zero algebra and $\sim$ be an equivalent relation on $\mathcal{A}$. They satisfy the following axioms:

A1. With every $a \in \mathcal{A}$ there is a complete lattice $(L_a, \leq)$ such that $L_a \subseteq \mathcal{A}$, where o and a are respectively the bottom and the top of L_a.

A2. For every $i \in L_a, [0,i]_{L_a} = \{x \in L_a | i \leq x \leq a\}$ is a sublattice of L_i.

A3. With every $i \in L_a$ there is an algebra denoted by a/i, such that $L_{a/i} = \{k/i | k \in [i,a]_{L_a}\}$ and the map given by $k \to k/i$ is an isomorphism between $[i,a]_{L_a}$ and $L_{a/i}$.

A4. If $a \sim b$, then there is an isomorphism $f: L_a \to L_b$ such that for every $l \in L_a, l \sim f(l)$ and $a/l \sim b/f(l)$.

A5. For every $a \in \mathcal{A}$ and $i,j \in L_a, (i \vee j/i) \sim (j/i \wedge j)$, and if $j \leq i$, then $(a/j)/(i/j) \sim a/i$.

Definition 2.1[1] *We say that an $i \in L_a$ is distinguished, which is denoted $i\triangle'a$, if $j = i = k$ for every $j,k \in L_a$ such that $j \leq i \leq k, i/j \sim k/i$ and i/j is a travial algebra.*

Definition 2.2[2] *The class $\mathcal{A}$ is called normal if it satisfies*

A6. If $a \in L_b$ and $i\triangle'a$, then $i \in L_b$.

In [1], [3] it was shown that the above classes $\mathcal{F}$, φ, g and β are all normal class. We can easily prove that b is also normal class (cf. [4]).

Remark Let $\mathcal{N}$ be the class of all near-rings. For every $A \in \mathcal{N}$, let L_A be the Lattice of all ideals of A and define $A \sim B$ if and only if they are isomorphic. Then $\mathcal{N}$ satisfies axiom A1-A5. But $\mathcal{N}$ is not normal class (cf. [5]).

Definition 2.3[1] *A subclass R of $\mathcal{A}$ is called a radical class if*

(I) For every $i \in L_a$, if $a \in R$, then $a/i \in R$;

(II) Every algebra a contains a largest (with respect to the order of L_a) R-ideal $R(a)$ called R-radical of a;

(III) For every $a \in \mathcal{A}$, $R(a/R(a)) = 0$.

For every radical class R the class $PR = \{a \in \mathcal{A} | R(a) = 0\}$ is called the semisimple class of R; elements of PR are called R-semisimple algebras.

Throughout the paper $\mathcal{A}$ is always assumed to be a normal class and all radical classed and algebras considered are supposed to be contained in $\mathcal{A}$. Our notation and terminology, unless otherwise state, agree with that of [1]. For the concepts of the lattice theory we refered the reader to any standard text such as [6].

Proposition 2.1 *Let R be an arbitrary radical class. If $i \in L_a$, then $R(i) \in L_a$.*

Proof Let $j,k \in L_i$ and $j \leq R(i) \leq k, R(i)/j \sim k/R(i)$. By definition, $R(i)$ and $R(i)/j \in R$. Since R is abstract, $k/R(i) \in R$. It follows that $k \in R$ by [1, Theorem 1]. Thus we obtain that $j = R(i) = k$. So $R(i)\triangle i$. But $\mathcal{A}$ is normal. Hence $R(i) \in L_a$.

Proposition 2.2 *Let R be an arbitrary radical class. If $a \in PR$ and $i \in L_a$, then $i \in PR$.*

Proof By proposition 2.1, $R(i) \in L_a$. By axiom A5 we have $R(i) \vee R(a)/R(i) \sim R(a)/R(i) \wedge R(a) \in R$. Hence, by [1, Theorem 1] $R(a) \leq R(i) \vee R(a) \in R$. Since $R(a)$ is the largest R-ideal of $a, R(a)) = R(i) \vee R(a)$. So $R(i) \leq R(a) = 0$. Thus $R(i) = 0$, i. e. $i \in PR$.

Proposition 2.3 *If $a/i \in PR$, then $R(a) \leq i$.*

Proof By axiom A5, $R(a) \vee i/i \sim R(a)/i \wedge R(a) \in R$. Since $R(a) \vee i/i \in L_{a/i}$, by proposition 2.2 $R(a) \vee i/i \in PR$. Consequently $R(a) \vee i/i \in R \bigcap PR = \{0\}$, which gives that $R(a) \leq R(a) \vee i = i$.

Now let R be an arbitrary radical class and a be an arbitrary algebra. For every $i \in L_a$ we denote by $\hat{i}$ the ideal of a, uniquely determined by $R(a/i) = \hat{i}/i$ in the sense of axiom A3. We have

Proposition 2.4 *Let a be an arbitrary algebra and $i, i_1, i_2 \in L_a$. Then*

(1) $\hat{\hat{i}} = \hat{i}$;

(2) if $i_1 \leq i_2$, *then* $\hat{i}_1 \leq \hat{i}_2$;

(3) $(\widehat{i_1 \vee i_2}) = (\widehat{\hat{i}_1 \vee \hat{i}_2})$.

Proof (1) By axiom A5, $a/\hat{i} \sim a/i/\hat{i}/i = a/i/R(a/i) \in PR$. So $0 = R(a/\hat{i}) = \hat{\hat{i}}/\hat{i}$ implies that $\hat{\hat{i}} = \hat{i}$.

(2) By axiom A5, we have the equivalence

$$\hat{i} \vee \hat{i_2}/\hat{i_2} \sim \hat{i_1}/(\hat{i_1} \wedge \hat{i_2}) \sim (\hat{i_1}/i_1)/(\hat{i_1} \wedge \hat{i_2}/i_1) = R(a/i_1)/(\hat{i_1} \wedge \hat{i_2}/i_1).$$

It follows that $\hat{i_1} \vee \hat{i_2}/\hat{i_1} \in R$ by [1, Theorem 1].

On the other hand, by axiom A3 we have

$$a/\hat{i_2} \sim (a/i)/(\hat{i_2}/i_2) = a/i_2/R(a/i_2) \in PR.$$

But $\hat{i_1} \vee \hat{i_2}/\hat{i_2} \in L_{a/\hat{i_2}}$. By virtue of Proposition 2.2, $\hat{i_1} \vee \hat{i_2}/\hat{i_2} \in PR$. Thus one gets that $\hat{i_1} \vee \hat{i_2}/\hat{i_2} = 0$. Hence $\hat{i_1} = \hat{i_1} \wedge \hat{i_2} \leq \hat{i_2}$ as desired.

(3) By $i_1 \vee i_2 \leq \hat{i_1} \vee i_2$ and by (2), $(\widehat{i_1 \vee i_2}) \leq \hat{i_1} \vee \hat{i_2}$. But $\hat{i_1} \vee \hat{i_2} \leq (\widehat{i_1 \vee i_2})$. We also have $(\widehat{\hat{i_1} \vee \hat{i_2}}) \leq (\widehat{\widehat{i_1 \vee i_2}}) = (\widehat{i_1 \vee i_2})$. So $(\widehat{i_1 \vee i_2}) = (\widehat{\hat{i_1} \vee \hat{i_2}})$.

3. General hereditary radicals

Definition 3.1 *A radical class R is called hereditary radical class if for every $a \in R$ and every $i \in L_a$, we have $i \in R$.*

Theorem 3.1 *The following statements about a radical class R are equivalent:*

(1) R is hereditary radical class;

(2) For any $i \in L_a, R(i) = i \wedge R(a)$;

(3) $R_a = \{R(i)|i \in L_a\}$ is a convex sublattice of L_a;

(4) $R_a = \{R(i)|i \in L_a\}$ is a convex subset of L_a.

Proof (1)⇒(2) By Proposition 2.1, $R(i) \in L_a$. Hence $R(i) \leq R(a) \wedge i \in [0, R(a)]_{L_a}$.

By axiom A2, $R(a) \wedge i \in L_{R(a)}$. So $R(a) \wedge i \in R$ by (1). Further, $R(a) \wedge i \leq R(i)$ since $R(a) \wedge i \in L_i$. Thus we get $R(i) = i \wedge R(a)$.

(2)⇒(3) Take any $j_1, j_2 \in R_a$. Then $j_1 = R(j_1) \in R$ and $j_2 = R(j_2) \in R$. In view of Proposition 2.1 $R_a \subseteq L_a$, it follows that $j_1 \leq R(a)$ and $j_2 \leq R(a)$. By (2), $j_1 \wedge j_2 = (j_1 \wedge j_2) \wedge R(a) = R(j_1 \wedge j_2) \in R$. Also note that $j_1 \vee j_2/j_1 \sim j_2/j_1 \wedge j_2 \in R$ and that R is closed under extensions, we have that $j_1 \vee j_2 \in R$. Moreover, if $j \in L_a$ and $j_1 \leq j \leq j_2$, then $j \in [0, j_2]_{L_a}$. Hence $j \in L_{j_2}$. By (2), $R(j) = j \wedge R(j_2) = j \wedge j_2 = j \in R_a$. Thus we obtain that R_a is a convex sublattice of L_a.

(3)⇒(4) It is trivial.

(4)⇒(1) Take any $a \in R$ and any $i \in L_a$. Then $R(i) \leq i \leq a = R(a)$. Since $R(i), R(a) \in R(a)$ and R_a is convex subset of $L_a, i \in R_a$. So $i = R(i) \in R$.

Theorem 3.2 *Let R be a hereditary radical class. If $i_1, i_2 \in L_a$, then $\widehat{(i_1 \wedge i_2)} = \hat{i}_1 \wedge \hat{i}_2$.*

Proof The theory was proved for associative rings in [2, Theorem 8.1]. The proof carries over mutatis mutandis for algebras.

Theorem 3.3 *For a radical class R the following statements are equivalent:*

(1) For every algebra $a \in \mathcal{A}$ and $i_1, i_2 \in L_a, R(i_1 \wedge i_2) = R(i_1) \wedge (i_2)$;

(2) For every algebra $a \in \mathcal{A}$, $I_a = \{i | i \in L_a \bigcap R\}$ is a sublattice of L_a.

Proof (1)⇒(2) Take any $i_1, i_2 \in I_a$. Then $i_1 \vee i_2/i_2 \sim i_1/i_1 \wedge i_2 \in R$. Further, $i_1 \vee i_2 \in R, i_1 \wedge i_2 = R(i_1) \wedge R(i_2) = R(i_1 \wedge i_2) \in R$. Hence $i_1 \wedge i_2, i_1 \vee i_2 \in I_a$, that is I_a is a sublattice of L_a.

(2)⇒(1) For arbitrary $a \in \mathcal{A}$ and any $i_1, i_2 \in L_a$, since $i_1 \wedge i_2 \in [0, i_1]_{L_a}$, $i_1 \wedge i_2 \in [0, i_2]_{L_a}$, we have $i_1 \wedge i_2 \in L_{i_1}$ and $i_1 \wedge i_2 \in L_{i_2}$ by Axiom A2. In virtue of proposition 2.1, $R(i_1 \wedge i_2) \leq R(i_1)$ and $R(i_1 \wedge i_2) \leq R(i_2)$. Hence $R(i_1 \wedge i_2) \leq R(i_1) \wedge R(i_2)$. On the other hand, it is easy to see that $R(i_1) \wedge R(i_2) \leq i_1 \wedge i_2$. By (2), we have $R(i_1) \wedge R(i_2) \leq R(i_1 \wedge i_2)$. So $R(i_1 \wedge i_2) = R(i_1) \wedge R(i_2)$.

Corollary 3.1 *If R is a hereditary radial class, then $R(i_1 \wedge i_2) = R(i_1) \wedge R(i_2)$ holds for any $a \in \mathcal{A}$ and any $i_1, i_2 \in L_a$.*

Proof Take any $i_1, i_2 \in I_a$. Since R is hereditary, $i_1 \wedge i_2 \in I_a$. Also, $i_1 \vee i_2/i_1 \sim i_2/i_1 \wedge i_2 \in R$ implies that $i_1 \vee i_2 \in I_a$. The result follows from Theorem 3.3.

4. General strongly semisimple radicals

Definition 4.1 *A radical class R is said to be strongly semisimple if for every $a \in PR$ and every $i \in L_a$, we always have $a/i \in PR$.*

Theorem 4.1 *The following statements about a radical class are equivalent:*

(1) R is strongly semisimple radical class;

(2) For every algebra a and every $i \in L_a$, if $i \geq R(a)$ then $a/i \in PR$;

(3) For every algebra a and every $i \in L_a, a/R(a) \vee i \in PR$;

(4) For every algebra a and every $i \in L_a, \hat{i} = R(a) \vee i$;

(5) For every algebra a and any $i_1, i_2 \in L_a, i_1 \vee i_2/i_1 \vee R(i_2) \in PR$;

(6) $\hat{I} = \{\hat{i} | i \in L_a\}$ is a convex subset of L_a.

Proof (1)⇒(2) Let $i \geq R(a)$. Since $a/i \sim (a/R(a))/(i/R(a))$ and $a/R(a) \in PR$, by (1) $a/i \in PR$.

(2)⇒(3) By $R(a) \vee i \geq R(a)$, it is clear.

(3)⇒(4) Since $a/R(a) \vee i \sim (a/i)/(R(a) \vee i/i) \in PR$, by Proposition 2.3 $R(a/i) \leq R(a) \vee i/i \in L_{a/i}$. But $R(a) \vee i/i \sim R(a)/R(a) \wedge i \in R$. Hence $R(a) \vee i/i \leq R(a/i)$. Thus $R(a/i) = R(a) \vee i/i$, $i \cdot e, \hat{i} = R(a) \vee i$.

(4)⇒(5) Since L_a is a modular lattice, we have $i_2 \wedge (i_1 \vee R(i_2)) = R(i_2) \vee (i_1 \wedge i_2)$. By $i_1 \vee (i_1 \vee R(i_2))$, we obtain that

$$i_1 \vee i_2/i_1 \vee R(i_2) \sim i_2/i_2 \wedge (i_1 \vee R(i_2)) = i_2/R(i_2) \vee (i_1 \wedge i_2).$$

But $i_2 \in \mathcal{A}$. By (4) we have $R(i_2/R(i_2 \vee (i_1 \wedge i_2)) = R(i_2 \vee (R(i_2) \vee (i_1 \wedge i_2))/R(i_2) \vee (i_1 \wedge i_2) = 0$. This implies that $i_1 \vee i_2/i_1 \vee R(i_2) \in PR$.

(5)⇒(1) Take any $a \in PR$ and any $i \in L_a$. Put $i_1 = i$ and $i_2 = a$. Then by (5) one gets $i \vee a/i \vee R(a) = a/i \in PR$. This shows that R is strongly semisimple radical class.

(4)⇒(6) Take $\hat{i_1}, \hat{i_2} \in \hat{I}$ and any $i \in L_a$ such that $\hat{i_1} \leq i \leq \hat{i_2}$. Since $a/\hat{i_1} \sim a/i_1/\hat{i_1}/i_1 = a/i/R(a/i) \in PR$, by Proposition 2.3, $R(a) \leq \hat{i_1}$. Thus, by (4) $\hat{i} = i \vee R(a) = i \in \hat{I}$.

(6)⇒(1) Take any $a \in PR$ and any $i \in L_a$. Then $o = R(a) = R(a/o) = \hat{o}/o, \hat{o} \in \hat{I}$. Also by $o = R(o) = R(a/a) = \hat{a}/a$ we know that $\hat{a} \in \hat{I}$. Since $o \leq i \leq a$ and $\hat{I}$ is a convex subset of $L_a, i \in \hat{I}$. Hence there exists an $i_1 \in L_a$ such that $\hat{i_1} = i$. Thus $a/i = a/\hat{i_1} \sim a/i_1/\hat{i_1}/i_1 = a/i/R(a/i) \in PR$, and the proof is complete.

Corollary 4.1 *Let a be an arbitrary algebra and $i_1, i_2 \in L_a$. If R is a strongly semisinple radical class, then*

$$R(i_1 \vee i_2) = R(i_1) \vee R(i_2), \widehat{(i_1 \vee i_2)} = \hat{i_1} \vee \hat{i_2}.$$

Proof By Theorem 4.1(5), we have $i_1 \vee i_2/R(i_1) \vee i_2 \in PR$ and $R(i_1) \vee i_2/R(i_1) \vee R(i_2) \in PR$. Moreover, $(i_1 \vee i_2/R(i_1) \vee R(i_2))/(R(i_1) \vee i_2/R(i_1) \vee R(i_2)) \sim i_1 \vee i_2/R(i_1) \vee i_2 \in PR$. Since PR is closed under extensions[1], $i_1 \vee i_2/R(i_1) \vee R(i_2) \in PR$. Thanks to Proposition 2.3, $R(i_1 \vee i_2) \leq R(i_1) \vee R(i_2)$. On the other hand, it is easy to see that $R(i_1) \vee R(i_2) \leq R(i_1 \vee i_2)$. Consequently $R(i_1 \vee i_2) = R(i_1) \vee R(i_2)$.

Also, by Theorem 4.1(4), $\widehat{(i_1 \vee i_2)} = (i_1 \vee i_2) \vee R(a) = (i_1 \vee R(a)) \vee (i_2 \vee R(a)) = \hat{i_1} \vee \hat{i_2}$.

Theorem 4.2 *For a radical class R the following statements are equivalent:*

(1) For any algebra a and $i_1, i_2 \in L_a, \widehat{(i_1 \vee i_2)} = \hat{i_1} \vee \hat{i_2}$;

(2) For any algebra a, $\hat{I} = \{\hat{i} | i \in L_a\}$ is a sublattice of L_a.

Proof (1)⇒(2) Take any $i_1, i_2 \in L_a$. by (1), $\hat{i_1} \vee \hat{i_2} = \widehat{(i_1 \vee i_2)} \in \hat{I}$. Moreover, $a/\hat{i} \in PR$ and $a/\hat{i} \in PR$. Put $j = \hat{i_1} \wedge \hat{i_2}$ and $R(a/j) = k/j$, where $k \in [j, a]_{L_a}$ by axiom A3. Then we have $k \vee \hat{i_1}/\hat{i_1} \sim k/k \wedge \hat{i_1}$ and $k \wedge \hat{i_1} \geq k \wedge j = j$. Accordingly $k/k \wedge \hat{i_1} \sim k/j/k \wedge \hat{i_1}/j \in R$. By Proposition 2.2, $k \vee \hat{i_1}/\hat{i_1} \in PR$. Consequently $k \vee \hat{i_1}/\hat{i_1} = 0$, and this means that $k \leq \hat{i_1} = k \vee \hat{i_1}$. Similarly we also have $k \leq \hat{i_2}$. Thus, $k \leq \hat{i_1} \wedge \hat{i_2} = j$. Hence $R(a/\hat{i_1} \wedge \hat{i_2}) = 0$. This implies that $\hat{i_1} \wedge \hat{i_2} = \widehat{(\hat{i_1} \wedge \hat{i_2})} \in \hat{I}$. So $\hat{I}$ is a sublattice of L_a.

(1)⇒(2) Take any $i_1, i_2 \in L_a$. By (2), $\hat{i_1} \vee \hat{i_2} \in \hat{I}$. Hence there exists $i \in L_a$ such that $\hat{i} = \hat{i_1} \vee \hat{i_2}$. Thanks to Proposition 2.4(1), $(\widehat{\hat{i_1} \vee \hat{i_2}}) = \hat{i_1} \vee \hat{i_2}$. On the other hand, by $i_1 \vee i_2 \leq \hat{i_1} \vee \hat{i_2}$ and by Proposition 2.4(2), we have $(\widehat{i_1 \vee i_2}) \leq (\widehat{\hat{i_1} \vee \hat{i_2}})$. But $\hat{i_1} \vee \hat{i_2} \leq (\widehat{i_1 \vee i_2})$. Hence $(\widehat{\hat{i_1} \vee \hat{i_2}}) \leq (\widehat{\widehat{i_1 \vee i_2}}) = (\widehat{i_1 \vee i_2})$. It follows that $(\widehat{i_1 \vee i_2}) = \hat{i_1} \vee \hat{i_2}$.

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代数正规类中的遗传根与强半单根

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摘 要: Puczylowski[1] 建立了一般代数对象类的根理论. 本文在代数正规类中，用格论方法刻划一般遗传根和强半单根类，探究它们的一些性质，推广了已知各类代数系统的某些根论研究.

关键词: 代数；正规类；格；遗传根；强半单根.