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On the Projective Radicals of Regular Rings *

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Abstract: We study the properties of projective radicals of regular rings. It is shown that the projective radical of a regular ring is left-right symmetric and a regular ring modulo its projective radical has zero projective radical. Also, we obtain a relation between projective radicals of a finitely generated projective module over a regular ring and its endomorphism ring, from which we give formulas about projective radicals of matrix rings and corners of a regular ring, and some equivalent conditions for a regular ring with zero projective radicals are given.

Key words: projective radical; socle; regular rings; MP-dimension

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1. Introduction

Given a right $R-$ module X, a submodule of X, Y, is called a maximal quotient projective submodule of X if X/Y is simple and projective. The projective radical (denoted by $P(X)$) of X is defined as the intersection of all maximal quotient projective submodules (if there is no any such submodules, then set $P(X) = X$). An equivalent definition for $P(X)$ is that

$$P(X) = \cap\{\mathrm{Ker} f \mid f \in \mathrm{Hom}(M, S), S \text{ is simple and projective}\}.$$

A module X is called to be meta-projective if $P(X) = 0$. The concept of projective radicals was first introduced in [1] and the authors used it to study the structure of modules. In [2], the commutative rings with $P(R) = 0$ were investigated and some characterizations for those rings were obtained. But when we discuss the properties of the projective radicals over an unrestricted ring, some difficulties occur. For example, although we can prove the projective radicals of any ring is always two-sided, they are not left-right sysmmetric in general as the following example shows:

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Example 1 Set $R = \begin{pmatrix} \mathbf{Z}_2 & \mathbf{Z}_2 \\ 0 & \mathbf{Z}_2 \end{pmatrix}$. Since R contains only eight elements, we can get the fact that $P(R_R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}$ and $P(R_R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\}$, by straightforward computations.

A ring is called to be (von Neumann) regular if for every $x \in R$, there exists $y \in R$ such that $x = xyx$. The class of regular rings is very important and is studied extensively. For details, one may refer to [3]. We all know the Jacobson radical of any regular rings is always zero, hence it is interesting to study other radicals of regular rings. The aim of this paper is to make a systematic study of the projective radicals for regular rings.

Throughout this paper, all rings are associative with nonzero identity and all modules are unital. An idempotent in R is called to be primitive if it can not be written as a sum of two nonzero orthogonal idempotents. The set of all primitive idempotents of R is denoted by $\pi(R)$. As usual, we denote the socle of a module M by $\mathrm{soc}(M)$ and use notations ι and γ to represent left and right annihilators respectively.

We conclude this section by recording some results in [1,2], which we need in the later section.

Lemma 1.1[1,Proposition1.5] *Let $f : M \mapsto N$ be a module homomorphism. Then $f(P(M)) \subseteq P(N)$.*

Lemma 1.2[1,Proposition1.6] $P(\bigoplus_{i=1}^n M_i) = \bigoplus_{i=1}^n P(M_i)$.

Lemma 1.3[2,Lemma1.1] *Let M_i be a family of meta-projective R-modules. Then both $\prod M_i$ and $\bigoplus M_i$ are meta-projective.*

2. Projective radicals

We first give a characterization for a simple and projective module over a regular ring, of which the proof is easy and omitted.

Lemma 2.1 *Let M be a right R-module over a regular ring R. Then the following statements are equivalent:*

(1) M is simple and projective;

(2) There exists $e \in \pi(R)$ such that $M \cong eR$.

Proposition 2.2 *Let R be a regular ring.*

(1) $P(R_R) = \cap_{e\in\pi(R)}(1-e)R = \gamma(\pi(R))$;

(2) $P({}_RR) = \cap_{e\in\pi(R)}R(1-e) = \iota(\pi(R))$.

Proof It is only necessary to prove (1). If A_R is a maximal quotient projective submodule of R_R, then A_R is a direct summand of R_R and so $A = fR$ for some idempotent f in R. Since $(1-f)R \cong R/A$ is simple and projective, we have $1-f \in \pi(R)$. Set $e = 1-f$, then $A = (1-e)R$ and $e \in \pi(R)$. Conversely, if $e \in \pi(R)$, then $(1-e)R$ is a maximal quotient projective submodule. Thus the first equation of (1) holds. The second equation is obvious. □

As is well known, if R is a regular ring, then two Loewy chains of R coincide. In particular, $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$ for any regular ring R. In the following, we use $\mathrm{soc}(R)$ to represent $\mathrm{soc}(R_R)$ and $\mathrm{soc}({}_RR)$.

Proposition 2.3 *Let R be a regular ring.*

(1) $P(R_R) = \gamma(\mathrm{soc}(R))$;

(2) $P({}_RR) = \iota(\mathrm{soc}(R))$.

Proof (1)

$$P(R_R) = \{x \in R | ex = 0 \text{ for all } e \in \pi(R)\} = \{x \in R | (\sum_{e\in\pi(R)} Re)x = 0\}$$
$$= \{x \in R | \mathrm{soc}(R)x = 0\} = \gamma(\mathrm{soc}(R)).$$

(2) is a dual to (1). □

Now, we obtain the main theorem of this section.

Theorem 2.4 *Let R be a regular ring. Then $P(R_R) = P({}_RR)$.*

Proof Assume $x \in P(R_R)$, then $\mathrm{soc}(R)x = 0$. Note that every (one-sided) ideal of a regular ring is idempotent, we have $x\mathrm{soc}(R) = x\mathrm{soc}(R)x\mathrm{soc}(R) = 0$ and so $x \in P({}_RR)$. Thus $P(R_R) \subseteq P({}_RR)$. Likewise, $P({}_RR) \subseteq P(R_R)$, and the result follows. □

Hereafter, the projective radical of a regular ring is denoted by $P(R)$. we list some examples to show that the projective radicals of regular rings hold all sorts of possibilities.

Example 2 Let F be a field and V an infinite-countably dimensional vecter space over F. Set $R = \mathrm{End}(V_F)$. Then R is a directly infinite regular ring. We will show that $P(R) = 0$. Let $v_i, i = 1, 2, \ldots$ be an F-base for V and define $e_i \in R, i = 1, 2, \ldots$ as $e_i(v_j) = \delta_{ij}v_j$. We see that $e_i \in \pi(R)$ for all i. If $x \neq 0$, then there exist nonzero elements $a_1, \ldots, a_n$ of F such that $a_1v_{i1} + \ldots a_nv_{in} \in xV$. Thus $a_1v_{i1} \in e_{i1}x(V)$ and so $e_{i1}x \neq 0$, whence $x \notin P(R)$. This implies $P(R) = 0$.

Example 3 Define R as above and let $J = \{x \in R | \dim xV < \infty\}$ be a maximal ideal of R. If $\bar{e}$ is a nonzero idempotent in R/J, then $\dim(eV) = \infty$ and eV has a direct decomposition $eV = V_1 \oplus V_2$ such that $\dim V_1 = \dim V_2 = \infty$. Hence $e = e_1 + e_2$ such that e_1, e_2 are orthogonal idempotents and $e_iV = V_i$ for $i = 1, 2$ and so $\bar{e}$ is not primitive in R/J, and it follows that $\mathrm{soc}(R/J) = 0$ and so $P(R/J) = R/J$.

Example 4 Define R and J as above and set $S = R \times R/J$. Noting that $\pi(S) = \{(e, 0) | e \in \pi(R)\}$, we immediately get $P(S) = R/J$.

Proposition 2.5 *Let R be a regular ring. Then $P(R/P(R)) = 0$.*

Proof First we claim that if $e \in \pi(R)$, then $\bar{e} \in \pi(R/P(R))$. Let $e \in \pi(R)$, then $e \in \mathrm{soc}(R)$, and $e\mathrm{soc}(R) \neq 0$, hence $\bar{e} \neq 0$; If $\bar{e} \notin \pi(R/P(R))$, then there exist nonzero orthogonal idempotents $\bar{e}_1, \bar{e}_2$ in $R/(P(R))$ such that $\bar{e} = \overline{e_1} + \overline{e_2}$. Note that $\overline{e_1} = \overline{e_1e}$ and $\overline{e_2} = \overline{e_2e}$, we have $e_1ee_2 \in P(R)$ and $e_2ee_1 \in P(R)$, and so $ee_1ee_2 = ee_2ee_1 = 0 \to \dagger$.

On the other hand, since $e-e_1-e_2 \in P(R)$, we have $e = ee_1+ee_2 = ee_1e+ee_2e$. In view of †, we get $(ee_1e)^2 = ee_1ee_1e = ee_1(e-ee_2)e = ee_1e$, and hence ee_1e, ee_2e are orthogonal idempotents in R. Thus either $ee_1e = 0$ or $ee_2e = 0$, and so either $\overline{e_1} = \overline{ee_1e} = 0$ or $\overline{e_2} = 0$, a contradiction.

Now if $\overline{x} \in P(R/(P(R))$, then $\overline{ex} = 0$ for all $e \in \pi(R)$. Thus $\mathrm{soc}(R)x \subseteq P(R)$ and it follows that $\mathrm{soc}(R)x = \mathrm{soc}(R)\mathrm{soc}(R)x = 0$. Hence $x \in P(R)$ and $\overline{x} = 0$. □

3. The computation of $P(R)$

Lemma 3.1 *Let M be a finitely generated right projective R-module over a regular ring R, and $\mathrm{End}(M_R) = S$. Then $P(M) = \cap\{\mathrm{Ker}f \mid f \in \pi(S)\}$.*

Proof Note that $f \in \pi(R)$ if and only if $f(M)$ is simple and projective, and the result follows in the same way as the proof of Proposition 2.2.

Theorem 3.2 *$P(M) = P(S)(M) \equiv \{f(m) \mid f \in P(S), m \in M\}$, with the above notations.*

Proof Let $g \in P(S)$ and $m \in M$. Since $fg = 0$ for any $f \in \pi(S)$, we have $g(m) \in \mathrm{Ker}f$ for every $f \in \pi(S)$ and it follows that $g(m) \in P(M)$ from Lemma 3.1. Conversely, given any $m \in P(M)$, then $f(m) = 0$ for all $f \in \pi(S)$. By [3,Theorem 1.11], there exists a submodule N of M such that $M = N \oplus mR$. Let $g : M \longmapsto mR$ be the natural projection, then $m = g(m)$ and $fg(M) = f(mR) = 0$ for all $f \in \pi(S)$. Hence $g \in P(S)$ and $m = g(m) \in P(S)(M)$. □

Corollary 3.3 *Let M be a finitely generated projective right module over a regular ring R with $P(R) = 0$. Then $P(\mathrm{End}(M_R)) = 0$.*

Corollary 3.4 *Let R be a regular ring.*

(1) $P(M_n(R)) = M_n(P(R))$ for any positive integer n,

(2) If e is an idempotent in R, then $P(eRe) = eP(R)e$.

Proof (1) Replacing M by nR, the direct sum of n copies of R, in Theorem 3.2, we have $nP(R) = P(M_n(R))(nR)$. Assume $A = (\alpha_1, \ldots, \alpha_n) \in P(M_n(R))$. Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, then $\alpha_i = Ae_i \in nP(R)$, and so $A \in M_n(P(R))$. Conversely, given any $A = (\alpha_1, \ldots, \alpha_n) \in M_n(P(R))$, then $\alpha_i \in nP(R)$ and there exist $B_i \in M_n(P(R))$ and $\beta_i \in nR$ such that $\alpha_i = B_i\beta_i$. Set $A_i = (0, \ldots 0, \alpha_i, 0, \ldots, 0)$ and $\Delta_i = (0, \ldots, 0, \beta_i, 0, \ldots, 0)$, then $A = A_1 + \ldots + A_n = B_1\Delta_1 + \ldots + B_n\Delta_n \in P(M_n(R))$.

(2) Note that $P(eR) = eP(R)$ and replace M by eR in Theorem 3.2, then we have $eP(R) = P(eRe)eR$. If $ere \in P(eRe)$, then $ere = ex$ for some $x \in P(R)$ and so $ere = exe \in eP(R)e$. For any $ere \in eP(R)e$ with $r \in P(R)$, we have $er \in P(eRe)eR$ and so $ere \in P(eRe)eRe = P(eRe)$, completing the proof. □

Let M be an R-module, recall the semi-reflexive radical (cf. [1]) of M is definded as $\cap\{\mathrm{Ker}f \mid f \in \mathrm{Hom}(M,R) \equiv M^*\}$ and denote it by $S(M)$.

Lemma 3.5 *Let R be a regular ring with $P(R) = 0$, and M a right R-module. Then $P(M) = S(M)$.*

Proof For any $f \in \mathrm{Hom}(M,R)$, let $t : eR \longmapsto R$ be the natural injection, then $tf \in M^*$

and Ker$f$ = Kertf. Thus $S(M) \subseteq P(M)$. Conversely, given any $f \in M^*$, let $p_e : R \longmapsto eR$ be the natural projections, then Ker$f = \cap\{\mathrm{Ker}(p_e f) | e \in \pi(R)\}$ by a straightfoward check. Hence $P(M) \subseteq S(M)$. □

Now, we give some equivalent conditions for a regular ring with $P(R) = 0$.

Theorem 3.6 *Let R be a regular ring, then the following statements are equivalent:*

(1) $P(R) = 0$;

(2) soc(R) is a essential right(left) ideal;

(3) $P(M) = S(M)$ for any right(left) R-module;

(4) There exists a faithful right(left) meta-projective R-module;

(5) Every right(left) projective module is meta-projective.

Proof (1)⇒(2) For any given $0 \neq x \in R$, $P(xR) = 0$ by Lemma 1.1 and so $xR = eR \oplus x_1 R$ for some $e \in \pi(R)$. Hence $xR \cap \mathrm{soc}(R) \supseteq eR \neq 0$.

(2)⇒(1) By Proposition 2.4, we have that $P(R)$ is a singular submodule of R_R and so $P(R) = 0$.

(1)⇒(3) This is Lemma 3.4.

(3)⇒(1) Since $S(R) = 0$, we get $P(R) = 0$ by (3).

(1)⇒(4) Note that R_R is a faithful right R-module.

(4)⇒(1) Let M be a faithful right meta-projective R-module, then R_R is isomorphic to a submodule of a direct product of copies of M. The result follows by Lemmas 1.1 and 1.3.

(1)⇒(5) This is an immediate consequence of Lemmas 1.1 and 1.3.

(5)⇒(1) This is trivial. □

Let R be a semisimple ring. Clearly, R is always regular and $P(R) = 0$. Example 2 shows that a regular ring with $P(R) = 0$ needs not be semisimple.

Proposition 3.7 *Let R be a regular ring with $P(R) = 0$. Then the following statements are equivalent:*

(1) R is semisimple;

(2) Every right meta-projective module is projective;

(3) Every right meta-projective module is injective.

Proof (1)⇒(2) and (1) ⇒ (3) are trivial.

(2)⇒(1) If X is a set, denote by R^X the direct product of X copies of R_R. By Lemma 1.3, we have $P(R^X) = 0$ and so R^X is projective for any set X. From a famous result of Chase in [4], it follows that R is right perfect and so $R = R/J(R)$ is semisimple.

(3)⇒(1) Given any right ideal J, we have $P(J_R) = 0$ and so J_R is injective, whence J_R is a direct summand of R. Hence R is semisimple. □

4. Closeness

Proposition 4.1 *Let R be a regular ring, and e be an idempotent in R. Then the following statements are equivalent:*

(1) $P(R) = 0$;

(2) $P(eRe) = 0$ and $P((1-e)R(1-e)) = 0$.

Proof (1)⇒(2) follows immediately from Corollary 3.3. (2) ⇒ (1): By Theorem 3.2, we have $P(eR) = 0$ and $P(1-e)R = 0$, and so $P(R) = P(eR) \oplus P((1-e)R) = 0$. □

Remark Set $e = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ in Example 1, then $eRe \cong (1-e)R(1-e) \cong \mathbf{Z}_2$ and so they are regular and have zero projective radicals. But we have seen that $P(R) \neq 0$. This should be contrasted with Proposition 4.1.

Proposition 4.2 *Let R be the direct product of regular rings R_i. Then $P(R) = 0$ if and only if $P(R_i) = 0$ for any i.*

Proof Note that $\pi(R) = \{e \in R | e_i \in \pi(R_i)$ for exactly one i, and $e_j = 0$ otherwise$\}$. Then the result follows easily. □

It is well-known that the subdirect product of semiprimitive rings (the rings with zero Jacoboson radicals) is semiprimitive. But this is not the case for the regular rings with zero projective radicals as the following example shows.

Example 5 Let R be the ring of all continuous maps from $\mathbf{Q}$, the set of rational numbers equipped with the topology inherited from $\mathbf{R}$, to $\mathbf{Z}_2$ with the discrete topology. Then R is a Boolean ring and so it is a subdirect product of copies of $\mathbf{Z}_2$. But we claim that $P(R) = R$. Let $0 \neq f \in R$, then $f^{-1}(1)$ is a nonempty open subset of $\mathbf{Q}$ and assume $x \in f^{-1}(1)$. Then there exist $a, b \in \mathbf{R}$ with $a < b$ such that $x \in (a,b) \cap \mathbf{Q} \subsetneq f^{-1}(1)$. Choose irrational numbers c, d such that $a < c < d < b$. By an easy observation, $(c,d) \cap \mathbf{Q}$ is a clopen set of $\mathbf{Q}$ and so we can define a contiuous map $g : \mathbf{Q} \longmapsto \mathbf{Z}_2$ by $g(y) = 1$ for every $y \in (c,d) \cap \mathbf{Q}$, otherwise $g(y) = 0$. Now $f = g + (f - g)$ and $g, f - g$ are nonzero orthogonal idempotents and we infer that $f \notin \pi(R)$. Hence $\mathrm{soc}(R) = 0$ and $P(R) = R$.

Recall that a right essential product of a collection $\{R_i\}$ of rings is any subdirect product of the R_i which contains an essential right ideal of the ring $\prod R_i$ (cf [5]).

Proposition 4.3 *Let R be a regular ring. If R is a right essential subdirect product of regular rings R_i with each $P(R_i) = 0$, then $P(R) = 0$.*

Proof Since $\mathrm{soc}(\prod R_i)$ is the intersection of all right essential ideals of $\prod R_i$, we have $\mathrm{soc}(\prod R_i) \subseteq R$. Noting that an idempotent e which is primitive in $\prod R_i$ is also primitive in R if $e \in R$, we infer that $\pi(\prod R_i) \subseteq \pi(R)$. So it follows that $P(R) = 0$ from Propositions 2.2 and 4.2. □

Finally, we give an example to show that the class of regular rings with $P(R) = 0$ is not closed under a direct limit.

Example 6 Let F be a field. Set $R_n = M_{2^n}(F)$ for all $n = 1, 2, \ldots$, which maps each $R_n \longmapsto R_{n+1}$ along the diagonal, i.e., map $x \longmapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, and set $R = \lim\limits_{\rightarrow} R_n$. Then R is a simple regular ring and so $\mathrm{soc}(R) = 0$ or $\mathrm{soc}(R) = R$. By [3, Example 8.1], R is not semisimple and so $\mathrm{soc}(R) = 0$. Hence $P(R) = R$.

5. MP-dimension

Assume R is a ring such that $P(_RR) = 0$. Note that every projective left $R-$module is meta-projective, so every left R-module has a meta-projective resolution. According to [2], the MP-dimension of A (denoted by $\text{MPdim}_R A$) is defined as $\inf\{n|$ there exists an exact sequence $0 \to P_n \to \ldots \to P_0 \to A \to 0$, where each P_i is meta-projective$\}$. The left MP-dimension of R (denoted by LMP-dimR) is defined as $\sup\{\text{MPdim}A | A$ is a left R-module$\}$. And RMP-dimR is defined similarly. Since every submodule of a meta-projective module is meta-projective, it is easy to see that LMP-dim$R = 0$ or 1.

In the case R is regular, we can decide when LMP-dim$R = 0$.

Theorem 5.1 *Let R be a regular ring such that $P(R) = 0$. Then the following statements are equivalent:*

(1) LMP-dim$R = 0$; *(2)* RMP-dim$R = 0$; *(3) every cyclic right R-module is meta-projective; (4) every cyclic left R-module is meta-projective; (5) R is semisimple.*

Proof It is only necessary to show (3)$\Rightarrow$ (5). Assume R is not semisimple. By Theorem 3.6 and [6, Proposition 5.10], we infer that $R/\text{soc}(R)$ is a nonzero singular right R-module. Since R is nonsingular, we have $\text{Hom}_R(R/\text{soc}(R), R) = 0$, by [5, Proposition 1.23]. From Lemma 3.5, it follows that $P(R/\text{soc}(R)) = R/\text{soc}(R)$, which contradictes (3).

Corollary 5.2 *Let R be a regular ring. Then* LMP-dim$R$=RMP-dim$R$.

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正则环的投射根

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摘 要: 研究了正则环上投射根的性质. 证明了正则环的投射根左右对称，且模去投射根的正则环只有零投射根. 给出了矩阵环及角落环投射根的计算式，并得到了投射根为零的正则环的一些刻画. 最后讨论了投射根为零的正则环在各种环运算下的封闭性和正则环的 MP- 维数.

关键词: 投射根；底座；正则环； MP- 维数.