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Unique Embeddings for Graphs on Orientable Surfaces Permitting Short Noncontractible Cycles *

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Abstract: A fundamental result in topological graph theory by H.Whitney states that a 3-connected graph has at most one planar embedding. C.Thomassen generalized this to LEW-embeddings on higher surfaces. We establish several unique embedding results for 3-connected graphs on orientable surfaces which admit relatively large facial walks and representativity and hence generalize Thomassen's uniqueness theorem on LEW-embeddings.

Key words: embedding; representativity; graph.

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1. Introduction

Graphs considered here are finite, simple, and undirected. Terminologies not explained may be found in Gross et al[1], Liu[3], and Thomassen[4].

A *surface* is a compact 2-manifold without boundary. An (A) *orientable* (*nonorientable*) surface of genus k is homeomorphic to the sphere with k *handles* (*crosscaps*). A graph G is called *embedded* on a surface Σ if G may be drawn on Σ such that each component of $\Sigma - E(G)$ is homeomorphic to an open disc and such embeddings are written as Π. For an embedding Π, its facial set is denoted by $\mathcal{F}$ and each face (facial walk) is called Π-*face* (*facial walk*). By a Π-facial walk f we sometime also mean a face of Π for convenience. If a facial walk has its boundary as a cycle (i.e., having no vertices repeated more than once), then it is also called a *facial cycle*. A curve (circuit) C on Σ is called *contractible* if $\Sigma - C$ is disconnected and one of the components, which Thomassen[4] called *inner part* of C, is homeomorphic to an open disc; otherwise it is named *noncontractible*. Let C be a cycle of a graph embedded on a surface. If C has no chord and $G - C$ has only one component, then it is called an *induced nonseparating cycle*. Two embeddings of

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G on Σ are considered to be the same if there exists a self-homeomorphism of the surface which induces an isomorphism of G. Let Π be an embedding of G on Σ and

$$ew(G,\Pi) = \min\{| C | : C \text{ is a noncontractible cycle of G on } \Sigma\},$$
$$\rho_\Sigma(G,\Pi) = \min\{| C \cap V(G) | : C \text{ is a noncontractible curve on } \Sigma\}$$

be respectively the *edge-width* and *representativity* (or *face-width* as some scholars called it) of Π. It is easy to see that $ew(G,\Pi) \geq \rho_\Sigma(G,\Pi)$. An embedding Π is called an *LEW-embedding* (i.e., *Large-Edge-Width-embedding*) if the length of every facial walk is less than $ew(G,\Pi)$. Intuitively, an LEW-embedding is one which embeds a given graph, if possible, densely. A fundamental result for topological graph theory by H.Whitney[6] states that a 3-connected graph has at most one planar embedding, i.e.

Theorem 1[6] *There is only one way to embed a 3-connected planar graph in the plane.*

W.Tutte[5] obtained Whitney's uniqueness theorem from a combinatorial view of facial walks—*induced nonseparating cycles*, i.e.,

Theorem 2[5] *Any planar 3-connected graph has a planar embedding such that every bounded region (i.e., facial walk) is convex.*

Thus, every isomorphism of a 3-connected graph on the plane can determine a homeomorphism of the sphere. To extend those to higher surfaces, C.Thomassen[4] investigated almost every aspect of LEW-embedding (a concept due to J.Hutchinson[2]) on general orientable surfaces and showed that such embeddings share many properties with 3-connected planar graphs.

Theorem 3[4] *Let Π be an LEW-embedding of a 3-connected graph G. Then the facial walks are precisely those of induced nonseparating cycles of G with length less than $ew(G,\Pi)$.*

Theorem 4[4] *For a 3-connected graph G and a given orientable surface Σ, there is at most one LEW-embedding.*

Although these results successfully extend Whitney's theories to higher surfaces in the case of LEW-embeddings, one may easily see that the conditions of Theorems 3 and 4 are somewhat too strict and hence many other unique embeddings for graphs on general surfaces may be excluded. For instance, many 3-connected near-triangulations (i.e., those having all facial walks 3-gons except possibly one which may have large length) with relative large representativity on a surface may also be embedded uniquely as analysed later. In fact, let

$$l = | \{f : f \in \mathcal{F}, | f | \geq ew(G,\Pi)\} |,$$

then LEW-embeddings are those of special case $l = 0$. We call such facial walks with length not less than the edge-width *large facial walks* (*cycles*). The purpose of this paper is to investigate the properties of the embeddings on surfaces which permit several relative large facial walks (with their length not less than $ew(G,\Pi)$) and hence fill some gaps left by Whitney, Tutte, and Thomassen.

Theorem A *Let G be a 3-connected graph with Π as an embedding on an orientable* *surface* Σ *such that* $\rho_\Sigma(G,\Pi) \geq \max\{l+1, 3\}$ *and*

$$\forall f \in \mathcal{F}, \mid f \mid \geq ew(G,\Pi) \Longrightarrow \mid f \mid < 2\rho_\Sigma(G,\Pi), \tag{1}$$

then G *is uniquely embedded on* Σ.

Remark (1) This generalizes Theorem 4 if considering the case of $l = 0$ since an LEW-embedding of a 3-connected graph satisfies condition (1) and further, it must have representativity ≥ 3 as we have shown late; (2) If a 3-connected graph G may be embedded on an orientable surface in different ways, at least one of the embeddings must have either several "large" facial walks or "small" representativity.

More generally, for a positive integer k, we have

Theorem B *Let* G *be a 3-connected graph with* Π *as an embedding on an orientable surface* Σ *such that* $\rho_\Sigma(G,\Pi) \geq \max\{(k-1)l+1, 3\}$ *and*

$$\forall f \in \mathcal{F}, \mid f \mid \geq ew(G,\Pi) \Longrightarrow \mid f \mid < k\rho_\Sigma(G,\Pi), \tag{2}$$

then G *is uniquely embedded on* Σ.

If we consider the case of $l = 1$ and ignore the restriction (2) on the lengths of large Π-facial walks, then we have the concept *NLEW-embedding* (i.e., *near-large-edge-width-embedding*) and the next

Theorem C *Let* G *be a 3-connected graph and* Σ *be an orientable surface. Then* (i)G has at most one NLEW-embedding with representativity ≥ 3 on Σ; (ii)G can not have both an LEW-embedding and an NLEW-embedding on the same orientable surface.

Remark (1) From the proof of Theorem C we may find that for a 3-connected graph G, it can not have both an LEW-embedding and an NLEW-embedding on the same orientable surface, but it may have both of them on distinct surfaces; (2) In [4] Thomassen showed that an LEW-embedding must also be a minimum embedding (i.e., the genus of the surface in which a 3-connected graph embedded is the genus of the graph). Thus if G is 3-connected and has both of an LEW- embedding and an NLEW-embedding, then the latter is not minimum.

Now we consider another version of Theorem B.

Theorem D *Let* G *be a 3-connected graph with* Π *as an embedding on an orientable surface* Σ *such that* $\rho_\Sigma(G,\Pi) \geq 3$ *and*

$$\forall f \in \mathcal{F}, \mid f \mid \geq ew(G,\Pi) \Longrightarrow (k-1)l < \mid f \mid < k\rho_\Sigma(G,\Pi), \tag{3}$$

where k *is a positive integer. Then* G *is uniquely embedded on* Σ.

Remark (1) One may see that Theorem B is not equivalent to Theorem D; (2) it seems that the low bound of Theorem D is sharp. One may see this from Fig.1, although it showing embedding on nonorientable surface, in which two combinatorial distinct embeddings of Peterson graph on the projective plane where $\rho = 3, l = 6, k = 2$ and the cycle $C = (a, b, c, d, e, f, a)$ in the left is transformed into a noncontractible one with exactly 6 large faces on the both sides of C.

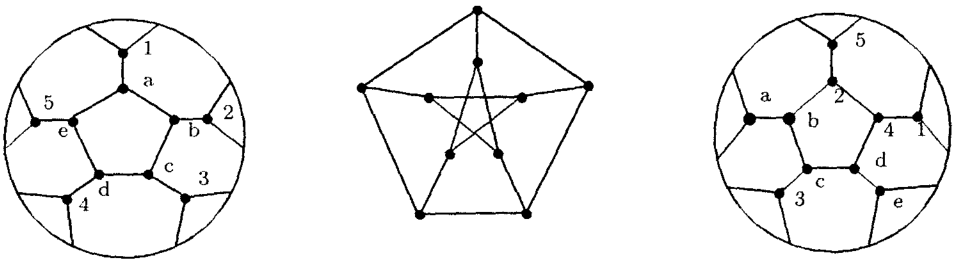

Fig.1. (2 distinct embeddings of Peterson Graph on the projective plane)

II. Proofs of main results

In this section we are concerned with the validity of Theorems A, B, C, and D. But first we should set up several lemmas.

Lemma 2.1 *Let G be a 2-connected graph with an embedding Π such that $\rho_\Sigma(G,\Pi) \geq 2$. Then each Π-facial walk is a cycle.*

Proof Suppose the contrary. Then Π has a facial walk, say f, which is not a cycle. Let C be a closed subwalk of f with shortest length. Then C is either a link edge or a cycle with length at least 2. Since G is 2-connected, C is either contractible or not. If C is noncontractible, then we have $\rho_\Sigma(G,\Pi) \leq 1$, a contradiction; if C is contractible, then C is the form $x_1x_2......x_kx_1$, where only x_1 is incident with the outer part of C (i.e., possible nonplanar part of $\Sigma - C$). Hence x_1 is a cutvertex, a contradiction as required. □

The following property says that the representativity of an LEW-embedding cannot be too small.

Lemma 2.2 *Let G be a 3-connected graph with an LEW-embedding Π on an orientable surface Σ, then $\rho_\Sigma(G,\Pi) \geq 3$.*

Proof Suppose the contrary. Then we have $\rho_\Sigma(G,\Pi) \leq 2$. If $\rho_\Sigma(G,\Pi) = 1$, then there exists a facial walk, say f, which contains a noncontractible cycle C of Σ intersecting f at only one vertex u as shown in Fig.2.

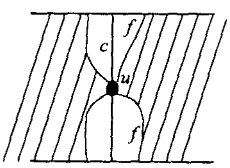

Fig. 2

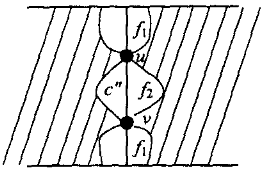

Fig.3

Let us consider the smallest closed subwalk of f, say C', which contains u. Since G is 2-connected and u is a repeated vertex of f, C' is a noncontractible cycle. Noting that $| f |< ew(G,\Pi)$, we have $| C' |< ew(G,\Pi)$, a contradiction with our definition of $ew(G,\Pi)$. If $\rho(\Pi) = 2$, then every facial walk is a cycle by Lemma 2.1 and there exists a noncontractible cycle, say C'', contained in the inner part of two facial cycles f_1, f_2 and intersecting G on the boundary of f_1, f_2 at u and v as shown in Fig.3. Note that u and v divide each boundary of f_1 and f_2 into two segments. Chose P_1 and P_2 as the shorter part of f_1 and f_2 (connecting u and v) respectively. Then

$$\max_{1\le i\le 2} | P_i |< \frac{1}{2}ew(G,\Pi),$$

and P_1 and P_2 form a noncontractible cycle C' with length

$$| C' |=| P_1 | + | P_2 |< ew(G,\Pi),$$

again contrary to the definition of $ew(G,\Pi)$. □

Lemma 2.3 *Let G be a 3-connected graph and Π be an embedding with representativity $\rho_\Sigma(G,\Pi) \geq 3$. Then every facial walk is an induced nonseparating cycle.*

Proof Let C be a facial cycle of Π. If C has a chord e with end vertices u and v on C, then e must be in the outer part of C since the inner part of C is an open disc. Let uPv be a segment on C determined by u and v . The cycle $uPv + e$ is contained in the boundaries of two faces. If $uPv + e$ is noncontractible, then we have $\rho_\Sigma(G,\Pi) \leq 2$; if $uPv + e$ is contractible, then $\{u, v\}$ will be a 2-cutvertex set since G has no multi-edges. Either case will lead to a contradiction. Hence C has no chord (i.e., an induced cycle). Next, suppose that $G-C$ has more than one component. Then some Π-facial cycle C' will intersect C at two vertices u' and v'. Again this will lead to a structure shown previously and consequently result in contradictions. This ends the proof of Lemma 2.3. □

Remark From the definition, the following holds:

Fact 1 If an induced nonseparating cycle is contractible in an embedding, then it bounds a facial walk.

Proof of Theorem A Suppose that G has two embeddings Π_1 and Π_2 on an orientable surface Σ and each has exactly l_1 and l_2 large facial cycles respectively, say f_i and $g_j (1 \leq i \leq l_1, 1 \leq j \leq l_2)$, such that for $1 \leq i \leq l_1, 1 \leq j \leq l_2$,

$$ew(G,\Pi_1) \leq| f_i |< 2\rho_\Sigma(G,\Pi_1), \quad ew(G,\Pi_2) \leq| g_j |< 2\rho_\Sigma(G,\Pi_2). \tag{4}$$

Note that by Euler's formula on polyhedrons, the number of Π_1-facial cycles is equal to those of Π_2. Let their facial sets be respectively as

$$\mathcal{F}_1 =\{f_1, f_2, \cdots, f_{l_1}, f_{l_1+1}, \cdots, f_m\},$$
$$\mathcal{F}_2 =\{g_1, g_2, \cdots, g_{l_2}, g_{l_2+1}, \cdots, g_m\}.$$

Suppose $ew(G,\Pi_1) \leq ew(G,\Pi_2)$. Then by definition each Π_1-facial cycle with length less than $ew(G,\Pi_1)$ is also a Π_2-facial cycle by Lemma 2.3, i.e.,

Fact 2 $\{f_{l_1+1}, \cdots, f_m\} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$ *and* $l_1 \geq l_2$.

By Lemma 2.3 each facial walk of Π_1 and Π_2 is an induced nonseparating cycle and further, we have

Fact 3 Π_1 has a facial cycle noncontractible in Π_2 if and only if Π_2 has a facial cycle noncontractible in Π_1.

Suppose that there is an large Π_2-facial cycle, say g_1, which is noncontractible in Π_1. By Lemma 2.1, no vertex appears more than once in the same facial cycle. If there is an large Π_1-facial cycle, say f_1, which is noncontractible in Π_2 and intersects g_1 at two segments P_1 and P_2, each having at least two vertices, then there exists a subpath P of f_1 which has no intermediate vertices of g_1 and joins two vertices u and v of g_1 (in fact we may chose P such that $\mid P \mid \leq ew(G, \Pi_1)$). Then a segment of g_1 determined by u and v together with P will form a cycle C. If C is contractible, then g_1 will be separable; if C is noncontractible, then each of the segments on g_1 determined by u and v will have at least $\rho_\Sigma(G, \Pi_1)$ vertices. Hence we have $\mid g_1 \mid \geq 2\rho_\Sigma(G, \Pi_1)$, thus (from (1)) we have $\rho_\Sigma(G, \Pi_2) > \rho_\Sigma(G, \Pi_1)$. Note that this argument is still available for f_1 (since f_1 is noncontractible in Π_2). By symmetry, $\rho_\Sigma(G, \Pi_1) > \rho_\Sigma(G, \Pi_2)$, a contradiction as required. We now have shown the following

Fact 4 If Π_2 has an large facial cycle which is noncontractible in Π_1, then any large Π_1-facial cycle which is noncontractible in Π_2 will intersect the former at no more than one segment which contains at least two vertices.

We now consider the case of "smaller" Π_2-facial cycles noncontractible in Π_1. Suppose that there is a Π_2- facial cycle, say g_{l_2+1}, with length less than $ew(G, \Pi_2)$ and noncontractible in Π_1. Then the length of it will not less than $\rho_\Sigma(G, \Pi_1)$ and by Fact 3, Π_1 has a large facial-cycle, say f_{l_2+1}, noncontractible in Π_2. Furthermore, we have

$$2\rho_\Sigma(G, \Pi_1) > \mid f_{l_2+1} \mid \geq ew(G, \Pi_1).$$

If those two cycles intersect each other at more than one segments, each having at least two vertices, then we have

$$2\rho_\Sigma(G, \Pi_1) \leq \mid g_{l_2+1} \mid, \quad 2\rho_\Sigma(G, \Pi_2) \leq \mid f_{l_2+1} \mid < 2\rho_\Sigma(G, \Pi_1),$$

i.e., $\rho_\Sigma(G, \Pi_1) > \rho_\Sigma(G, \Pi_2)$. Further, these will imply $ew(G, \Pi_2) > 2\rho_\Sigma(G, \Pi_2)$, a contradiction with the conditions of Theorem A. Thus, we have proved the following

Fact 5 If Π_2 has a facial cycle which is noncontractible in Π_1 and with length less than $ew(G, \Pi_2)$, then any large Π_1-facial cycle which is noncontractible in Π_2 will intersect the former at no more than one segment which has at least two vertices.

Suppose that Π_2 has a facial cycle, say g, noncontractible in Π_1. Then the length of g is at least $\rho_\Sigma(G, \Pi_1)$, i.e., there are at least $\rho_\Sigma(G, \Pi_1)(> l_1 \geq l_2)$ domains bound each side of the boundary of g. By Facts 4 and 5 and an easy counting one may see that there exit three Π_2-facial cycles (one of them is g) sharing a common edge, a contradiction with the fact that each edge in an embedding is on the boundary of at most two facial walks. Thus, each Π_2-facial cycle is also a Π_1-facial cycle. Since every edge is in two Π_1-facial cycles and two Π_1-facial cycles, we conclude that the Π_1-facial cycles are the same as the Π_2-facial cycles, i.e., $\mathcal{F}_1 = \mathcal{F}_2$. Hence $\Pi_1 = \Pi_2$. This completes the proof of Theorem A.

From the proof of Theorem A, we may conclude the next

Lemma 2.4 *Let G be a 3-connected graph with Π as an embedding on an orientable surface Σ. Suppose that f is an induced nonseparating noncontactible cycle and g is a facial cycle of Π. If f and g share k segments (each of them contains at least two vertices) in common, then*

$$|f| \geq k\rho_\Sigma(G, \Pi). \tag{5}$$

As a matter of fact, we may suppose that such k segments have respectively the form of $x_1P_1y_1, x_2P_2y_2, ..., x_kP_ky_k$ (where $x_i \neq y_i$ for $1 \leq i \leq k$). By Lemma 2.1,

$$x_iP_iy_i \cap x_jP_jy_j = \phi, \quad 1 \leq i \neq j \leq k.$$

Since every segment is on the boundary of g, there are k pairwise disjoint subpaths of g, say $Q_1, Q_2, \cdots, Q_k$, such that each Q_i connects two vertices of f and has no intermediate vertices on f. As we have reasoned in the proof of Theorem A, all three cycles formed by Q_i and f are noncontractible. Thus, each segment of f determined by the two ends of Q_i has length at least $\rho_\Sigma(G, \Pi)$. Hence (5) follows from counting of this for $1 \leq i \leq k$. □

Proof of Theorem B We may employ Lemma 2.4 to the procedure used in our proof of Theorem A and notice that Facts $1, \cdots, 5$ are available and finally conclude that (under the hypothesis in our proof of Theorem A) if Π_2 has a facial cycle, say g, noncontractible in Π_1, then the length of g is at least $\rho_\Sigma(G, \Pi_1)$, i.e., there are at least $\rho_\Sigma(G, \Pi_1)(> (k-1)l_1 \geq (k-1)l_2)$ domains bound the two sides of g. By Facts 4 and 5 and an easy counting one may see that there exit three Π_2-facial cycles (one of them is g) sharing a common edge, a contradiction with the fact that each edge in an embedding is on the boundary of at most two facial walks. Thus, each Π_2-facial cycle is also a Π_1-facial cycle. Since every edge is in two Π_1-facial cycles and two Π_1-facial cycles, we conclude that the Π_1-facial cycles are the same as the Π_2-facial cycles, i.e., $\mathcal{F}_1 = \mathcal{F}_2$. Hence $\Pi_1 = \Pi_2$. This completes the proof of Theorem B. □

Proof of Theorem C Let us consider the case of $l = 1$ in the proof of Theorem A and ignore the restriction $|f_1| \geq 2\rho(\Pi_1)$ ($|g_1| \geq 2\rho(\Pi_2)$). As we have reasoned before, all Π_2-facial cycles are Π_1-facial cycles except possibly g_1 (which is also an induced nonseparating cycle from Lemma 2.3). If g_1 is noncontractible in Π_1, then from $\rho_\Sigma(G, \Pi_2) \geq 3$ and Lemma 2.1 we may conclude that there are at least two Π_1-facial cycles which are also of Π_2 sharing a common edge of g_1, although g_1 is only possible non-Π_2-facial cycle. Again we will see a contradiction appearing in the previous paragraphs. Thus g_1 must be contractible in Π_1 and consequently be a Π_1-facial cycle. Since g is arbitrary and following the reasoning we have used in the end of last paragraph we see that $\Pi_1 = \Pi_2$. This ends the proof of Theorem C. □

Proof of Theorem D Let us start from the assumptions at the beginning of Theorem

A's proof. Notice that Lemmas $1, \cdots, 5$ are still valid and what we should do is to consider the case that there is a Π_2-facial cycle g which is noncontractible in Π_1. Now we have

$$(k-1)l_1 < \rho_\Sigma(G, \Pi_1) \leq |g|$$

domains lying on both sides of g. Since $l_1 \geq l_2$ and each large Π_1-facial cycle noncontractible in Π_2 intersects g at no more than $k-1$ segments, each has at least two vertices, by Facts 4, 5 and Lemma 2.4. Since the number of such large Π_1-facial cycles noncontractible in Π_2 is no larger than $l_1 (\geq l_2)$ and there are at least $(k-1)l_1 + 1$ Π_1-facial cycles on the both sides of g, there exist at least two Π_2-facial cycles which will share a common edge of g, a contradiction (with the fact that an edge in an embedding will be on the boundaries of at most two faces). This ends the proof of Theorem D. □

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可定向曲面上具有较短不可收缩圈图嵌入的唯一性

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摘 要: 拓扑图论中的一个基本问题就是要决定图在一个 (可定向) 曲面上的嵌入之数目 (既嵌入的柔性问题). H.Whitney 的经典结果表明: 一个 3- 连通图至多有一个平面嵌入; C.Thomassen 的 LEW- 嵌入 (大边宽度) 理论将这一结果推广到一般的可定向曲面. 本文给出了几个关于一般可定向曲面上嵌入图的唯一性定理. 结果表明: 一些具有大的面迹的可定向嵌入仍然具有唯一性. 这在本质上推广了 C.Thomassen 在 LEW- 嵌入方面的工作.

关键词: 嵌入; 表示数; 图.