Abstract Let $R$ be a ring. A right $R$-module $M$ is called $f$-projective if $\text{Ext}^1(M, N) = 0$ for any $f$-injective right $R$-module $N$. We prove that $(\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj})$ is a complete cotorsion theory, where $\mathcal{F}\text{-proj}$ ($\mathcal{F}\text{-inj}$) denotes the class of all $f$-projective ($f$-injective) right $R$-modules. Semihereditary rings, von Neumann regular rings and coherent rings are characterized in terms of $f$-projective modules and $f$-injective modules.

Keywords $f$-projective module; $f$-injective module; finitely presented cyclic module; (pre)envelope; (pre)cover.

1. Introduction

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. We use $M_R$ to indicate a right $R$-module, $\text{FI}(M_R)$ stands for the $f$-injective envelope of $M_R$, the character module $M^+$ is defined by $M^+ = \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$, and $\text{pd}(M_R)$ denotes the projective dimension of $M_R$. $\text{Hom}(M, N)$ ($\text{Ext}^n(M, N)$) denotes $\text{Hom}_R(M, N)$ ($\text{Ext}^n_R(M, N)$) for an integer $n \geq 1$. General background material can be found in Ref. [1, 6, 12, 15].

A module $M_R$ is called $f$-injective (or $\aleph_0$-injective; coflat) if $\text{Ext}^1_R(R/I, M) = 0$ for any finitely generated right ideal $I$ of $R$. $f$-injective modules have been studied in many papers such as Ref. [2-4, 8]. In Section two of this paper, we first introduce the notion of $f$-projective modules, and then give some equivalent characterizations of these modules when $R$ is a self $f$-injective ring. For instance, it is shown that if $R$ is self $f$-injective, then $M$ is $f$-projective if and only if $M$ is a cokernel of an $f$-injective preenvelope $K \to F$ with $F$ projective. We also prove that $(\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj})$ is a complete cotorsion theory, where $\mathcal{F}\text{-proj}$ ($\mathcal{F}\text{-inj}$) denotes the class of all $f$-projective ($f$-injective) right $R$-modules. In Section three, some new characterizations of semihereditary rings, von Neumann regular rings and coherent rings are given. For example, it is proven that $R$ is a right semihereditary ring if and only if every (quotient module of any injective) right $R$-module $M$ has a monic $\mathcal{F}\text{-inj}$-cover if and only if $\text{pd}(M) \leq 1$ for every $f$-projective right $R$-module $M$ if and only if $(\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj})$ is hereditary and every $f$-projective right $R$-module
has a monic \( F \)-inj-cover; \( R \) is a von Neumann regular ring if and only if every cotorsion right \( R \)-module is \( f \)-injective if and only if every \( f \)-projective right \( R \)-module is projective if and only if every \( f \)-projective right \( R \)-module is flat if and only if every right \( R \)-module has an \( F \)-inj-envelope with the unique mapping property if and only if \((F \text{-proj}, F \text{-inj})\) is hereditary and every \( f \)-projective right \( R \)-module has an \( F \)-inj-envelope with the unique mapping property if and only if \((F \text{-proj}, F \text{-inj})\) is hereditary and every \( f \)-projective right \( R \)-module is \( f \)-injective. Finally, as a generalization of the well known result that \( R \) is a right coherent ring if and only if every direct limit of \( FP \)-injective right \( R \)-modules is \( FP \)-injective, we get that \( R \) is a right coherent ring if and only if every direct limit of \( f \)-injective right \( R \)-modules is \( f \)-injective.

2. Definition and general results

We start with the following

**Definition 2.1** Let \( M \) be a right \( R \)-module. \( M \) is called an \( f \)-projective module if \( \text{Ext}^1(M, N) = 0 \) for any \( f \)-injective right \( R \)-module \( N \).

**Remark 2.2** Any finitely presented cyclic \( R \)-module is \( f \)-projective, and it is easily seen that all (left) right \( R \)-modules are \( f \)-projective if and only if ring \( R \) is (left)right Noetherian.

Recall that a pair \((F, C)\) of classes of right \( R \)-modules is called a cotorsion theory\([6]\) if \( F^\perp = C \) and \( \perp C = F \), where \( F^\perp = \{ C : \text{Ext}^1(F, C) = 0 \text{ for all } F \in F \} \), and \( \perp C = \{ F : \text{Ext}^1(F, C) = 0 \text{ for all } C \in C \} \).

Let \( C \) be a class of right \( R \)-modules and \( M \) a right \( R \)-module. A homomorphism \( \varphi : M \to F \) with \( F \in C \) is called a \( C \)-preenvelope of \( M \)\([6]\) if for any homomorphism \( f : M \to F' \) with \( F' \in C \), there is a homomorphism \( g : F \to F' \) such that \( g\varphi = f \). Moreover, if the only such \( g \) are automorphisms of \( F \) when \( F' = F \) and \( f = \varphi \), the \( C \)-preenvelope \( \varphi \) is called a \( C \)-envelope of \( M \).

Following Ref. [6, Definition 7.1.6], a monomorphism \( \alpha : M \to C \) with \( C \in C \) is said to be a special \( C \)-preenvelope of \( M \) if \( \text{coker}(\alpha) \in \perp C \). Dually we have the definitions of a (special) \( C \)-precovar and a \( C \)-cover. Special \( C \)-preenvelopes (resp., special \( C \)-precovers) are obviously \( C \)-preenvelopes (resp., \( C \)-precovers).

A \( C \)-envelope \( \varphi : M \to F \) is said to have the unique mapping property\([5]\) if for any homomorphism \( f : M \to F' \) with \( F' \in C \), there is a unique homomorphism \( g : F \to F' \) such that \( g\varphi = f \).

**Proposition 2.3** Let \( R \) be a right self \( f \)-injective ring and \( M \) a right \( R \)-module. Then the following are equivalent:

1. \( M \) is \( f \)-projective;
2. \( M \) is projective with respect to every exact sequence \( 0 \to A \to B \to C \to 0 \), where \( A \) is \( f \)-injective;
3. For every exact sequence \( 0 \to K \to F \to M \to 0 \), where \( F \) is \( f \)-injective, \( K \to F \) is an \( f \)-injective preenvelope of \( K \);
4. \( M \) is a cokernel of an \( f \)-injective preenvelope \( K \to F \) with \( F \) projective.
Proof  (1) ⇒ (2) is obvious.

(2) ⇒ (1). For every \( f \)-injective right \( R \)-module \( N \), there is a short exact sequence \( 0 \to N \to E \to L \to 0 \) with \( E \) injective, which induces an exact sequence \( \text{Hom}(M, E) \to \text{Hom}(M, L) \to \text{Ext}^1(M, N) \to 0 \). Since \( \text{Hom}(M, E) \to \text{Hom}(M, L) \to 0 \) is exact by (2), \( \text{Ext}^1(M, N) = 0 \). So (1) follows.

(1) ⇒ (3) is easy to verify.

(3) ⇒ (4). Let \( 0 \to K \to P \to M \to 0 \) be an exact sequence with \( P \) projective. Note that \( P \) is \( f \)-injective by hypothesis, thus \( K \to P \) is an \( f \)-injective preenvelope.

(4) ⇒ (1). By (4), there is an exact sequence \( 0 \to K \to P \to M \to 0 \), where \( K \to P \) is an \( f \)-injective preenvelope with \( P \) projective. It gives rise to the exactness of \( \text{Hom}(P, N) \to \text{Hom}(K, N) \to \text{Ext}^1(M, N) \to 0 \) for each \( f \)-injective right \( R \)-module \( N \). Note that \( \text{Hom}(P, N) \to \text{Hom}(K, N) \to 0 \) is exact by (4). Hence \( \text{Ext}^1(M, N) = 0 \), as desired.

Denote by \( \mathcal{F} \)-proj (\( \mathcal{F} \)-inj) the class of all \( f \)-projective (\( f \)-injective) right \( R \)-modules. Then we have

Theorem 2.4  Let \( R \) be a ring. Then \( (\mathcal{F} \)-proj, \( \mathcal{F} \)-inj) is a cotorsion theory. Moreover, every right \( R \)-module has a special \( \mathcal{F} \)-inj-preenvelope and every right \( R \)-module has a special \( \mathcal{F} \)-proj-precover.

Proof  Let \( X \) be the set of representatives of finitely presented cyclic right \( R \)-modules. Thus \( \mathcal{F} \)-inj= \( X \perp \). Since \( \mathcal{F} \)-proj= \( \perp (X \perp ) \), the results follow from Ref. [6, Definition 7.1.5] and [7, Theorem 10].

Remark 2.5  The statement of Theorem 2.4 is the best possible in the sense that \( (\mathcal{F} \)-proj, \( \mathcal{F} \)-inj) is not a perfect cotorsion theory because \( f \)-injective envelopes may not exist in general [14, Proposition 4.8]. However, if \( \mathcal{F} \)-proj is closed under direct limits, then \( (\mathcal{F} \)-proj, \( \mathcal{F} \)-inj) is a perfect cotorsion theory by Theorem 2.4 and Ref. [6, Theorem 7.2.6].

Corollary 2.6  The following are equivalent for a ring \( R \):

\begin{enumerate}
  \item Every right \( R \)-module is \( f \)-projective;
  \item Every cyclic right \( R \)-module is \( f \)-projective;
  \item Every \( f \)-injective right \( R \)-module is injective;
  \item \( (\mathcal{F} \)-proj, \( \mathcal{F} \)-inj) is hereditary, and every \( f \)-injective right \( R \)-module is \( f \)-projective.
\end{enumerate}

In this case, \( R \) is right Noetherian.

Proof  (1) ⇒ (2) is clear.

(2) ⇒ (3). Let \( N \) be any \( f \)-injective right \( R \)-module and \( I \) any right ideal of \( R \). Then \( \text{Ext}^1(R/I, N) = 0 \) by (2). Thus \( N \) is injective, as desired.

(3) ⇒ (1) holds by Theorem 2.4.

(1) ⇒ (4) is clear.

(4) ⇒ (1). By Theorem 2.4, for any right \( R \)-module \( M \), there is a short exact sequence \( 0 \to M \to F \to L \to 0 \), where \( F \) is \( f \)-injective and \( L \) is \( f \)-projective. So (1) follows from (4).
In this case, $R$ is right Noetherian since every $f$-injective right $R$-module is injective.

3. Applications

Recall that a ring $R$ is right semihereditary if and only if every finitely generated right ideal of $R$ is projective.

**Theorem 3.1** The following are equivalent for a ring $R$:

1. $R$ is a right semihereditary ring;
2. Every quotient module of any $(f)$-injective right $R$-module is $f$-injective;
3. Every (quotient module of any injective) right $R$-module $M$ has a monic $\mathcal{F}$-inj-cover $\varphi : F \to M$;
4. $\text{pd}(R/I) \leq 1$ for every right $R$-module $R/I$ with $I$ finitely generated right ideal of $R$;
5. $\text{pd}(M) \leq 1$ for every $f$-projective right $R$-module $M$;
6. $(\mathcal{F}$-proj, $\mathcal{F}$-inj) is hereditary, and every $f$-projective right $R$-module has a monic $\mathcal{F}$-inj-cover.

**Proof** (1) $\iff$ (2) holds by Ref. [8, Theorem 3.2].

(2) $\Rightarrow$ (3). Let $M$ be any right $R$-module. Write $F = \sum\{N \leq M : N \in \mathcal{F}$-inj$\}$ and $G = \oplus\{N \leq M : N \in \mathcal{F}$-inj$\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Note that $G \in \mathcal{F}$-inj, so $F \in \mathcal{F}$-inj by (2). Next we prove that the inclusion $i : F \to M$ is an $\mathcal{F}$-inj-cover of $M$. Let $\psi : F' \to M$ with $F' \in \mathcal{F}$-inj be an arbitrary right $R$-homomorphism. Note that $\psi(F') \subseteq F$ by (2). Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is an $\mathcal{F}$-inj-precover of $M$. In addition, it is clear that the identity map $1_F$ of $F$ is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (3) follows.

(3) $\Rightarrow$ (2). Let $M$ be any $f$-injective right $R$-module and $N$ any submodule of $M$. We shall show that $M/N$ is $f$-injective. Indeed, there exists an exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective. Since $L$ has a monic $\mathcal{F}$-inj-cover $\varphi : F \to L$ by (3), there is $\alpha : E \to F$ such that the following exact diagram is commutative:

\[
\begin{array}{ccccccccc}
0 & \to & N & \to & E & \to & L & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & F & \to & & & & \end{array}
\]

Thus $\varphi$ is epic, and hence it is an isomorphism. Therefore $L$ is $f$-injective. For any finitely presented cyclic right $R$-module $K$, we have

\[
0 = \text{Ext}^1(K, L) \to \text{Ext}^2(K, N) \to \text{Ext}^2(K, E) = 0.
\]

Therefore $\text{Ext}^2(K, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

\[
0 = \text{Ext}^1(K, M) \to \text{Ext}^1(K, M/N) \to \text{Ext}^2(K, N) = 0.
\]
Therefore \( \text{Ext}^1(K, M/N) = 0 \), as desired.

(2) \( \Rightarrow \) (5). Let \( M \) be any \( f \)-projective right \( R \)-module and \( N \) any right \( R \)-module. There exists an exact sequence \( 0 \to N \to E \to L \to 0 \) with \( E \) injective, and hence \( L \) is \( f \)-injective by (2), so we have the exact sequence

\[
0 = \text{Ext}^1(M, L) \to \text{Ext}^2(M, N) \to \text{Ext}^2(M, E) = 0.
\]

Therefore \( \text{Ext}^2(M, N) = 0 \), which implies \( \text{pd}(M) \leq 1 \).

(5) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (1) are clear.

(2) \( \Rightarrow \) (6) is clear by the equivalence of (2) and (3).

(6) \( \Rightarrow \) (2). Let \( M \) be any \( f \)-injective right \( R \)-module and \( N \) any submodule of \( M \). We have to prove that \( M/N \) is \( f \)-injective. In fact, note that \( N \) has a special \( \mathcal{F} \)-inj-preenvelope, i.e., there exists an exact sequence \( 0 \to N \to E \to L \to 0 \) with \( E \in \mathcal{F}\text{-Inj} \) and \( L \in \mathcal{F}\text{-Proj} \). The rest of the proof is similar to that of (3) \( \Rightarrow \) (2) by noting that \( \text{Ext}^2(K, E) = 0 \) for any finitely presented cyclic right \( R \)-module \( K \), since \((\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj}) \) is hereditary. This completes the proof.

A right \( R \)-module \( M \) is called cotorsion \[^{[6, \text{Definition} 5.3.22]} \] if \( \text{Ext}^1(F, M) = 0 \) for all flat right \( R \)-modules \( F \). It is well known that a ring \( R \) is von Neumann regular if and only if every (cyclic) right \( R \)-module is flat if and only if every cotorsion right \( R \)-module is flat if and only if every cotorsion right \( R \)-module is injective. Now we have

**Theorem 3.2** The following are equivalent for a ring \( R \):

1. \( R \) is a von Neumann regular ring;
2. Every right \( R \)-module is \( f \)-injective;
3. Every cotorsion right \( R \)-module is \( f \)-injective;
4. Every \( f \)-projective right \( R \)-module is projective;
5. Every \( f \)-projective right \( R \)-module is flat;
6. Every finitely presented cyclic right \( R \)-module is flat;
7. Every right \( R \)-module has an \( \mathcal{F} \)-inj-envelope with the unique mapping property;
8. \((\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj}) \) is hereditary and every \( f \)-projective right \( R \)-module has an \( \mathcal{F} \)-inj-envelope with the unique mapping property;
9. \((\mathcal{F}\text{-proj}, \mathcal{F}\text{-inj}) \) is hereditary and every \( f \)-projective right \( R \)-module is \( f \)-injective.

**Proof** (1) \( \iff \) (2) holds by Ref. [4, Proposition 1.11], (2) \( \Rightarrow \) (3) and (4) through (6) are obvious.

(3) \( \iff \) (6) follows from Ref. [10, Proposition 2.10].

(6) \( \Rightarrow \) (1). Note that \( R/I \) is flat for any finitely generated right ideal \( I \) of \( R \) by (6). Thus \( R/I \) is projective since \( R/I \) is finitely presented. It follows that \( I \) is a direct summand of \( R \), which implies that \( R \) is von Neumann regular.

(2) \( \Rightarrow \) (7) and (7) \( \Rightarrow \) (8) are clear.
(7)⇒(2). Let \( M \) be a right \( R \)-module. There is the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & M & \xrightarrow{\sigma M} & \text{FI}(M) & \xrightarrow{\gamma} & L & \xrightarrow{\sigma L \gamma} & \text{FI}(L) & 0 \\
& & \downarrow & & \downarrow & & \sigma L & & \downarrow & \\
& & 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

Note that \( \sigma L \gamma \sigma M = 0 = \sigma M \), so \( \sigma L \gamma = 0 \) by (7). Therefore \( L = \text{im}(\gamma) \subseteq \ker(\sigma L) = 0 \), and hence \( M \) is \( f \)-injective.

(8)⇒(9). Let \( M \) be an \( f \)-projective right \( R \)-module. By (8), there is an exact sequence

\[
0 \to M \xrightarrow{\sigma M} \text{FI}(M) \xrightarrow{\gamma} L \to 0,
\]

where \( L \) is \( f \)-projective by Wakamatsu’s Lemma[6, Proposition 7.2.4]. Thus \( M \) is \( f \)-injective by the proof of (7)⇒(2).

(9)⇒(2). Let \( M \) be any right \( R \)-module. Note that \( M \) has a special \( \mathcal{F} \)-proj-precovers, i.e., there exists an exact sequence \( 0 \to K \to L \to M \to 0 \) with \( K \in \mathcal{F}\text{-inj} \) and \( L \in \mathcal{F}\text{-proj} \). Thus \( L \in \mathcal{F}\text{-inj} \) and \( M \in \mathcal{F}\text{-inj} \) by (9).

By Ref. [13, Theorem 3.2], a ring \( R \) is right coherent if and only if every direct limit of \( \mathcal{F}P \)-injective \( R \)-modules is \( \mathcal{F}P \)-injective. We generalize this result as follows.

**Theorem 3.3** The following are equivalent for a ring \( R \):

1. Every factor module of an \( f \)-injective right \( R \)-module by a pure submodule is \( f \)-injective;
2. Every direct limit of \( f \)-injective right \( R \)-modules is \( f \)-injective;
3. \( R \) is a right coherent ring.

**Proof** (1)⇒(2). Let \( (M_i, f_{ij})_\Lambda \) be a direct system of \( f \)-injective right \( R \)-modules. Then \( \oplus M_i \) is \( f \)-injective by Ref. [8, Theorem 2.4], and the canonical epimorphism \( \oplus M_i \to \lim M_i \) is pure by Ref. [15, 33.9]. So \( \lim M_i \) is \( f \)-injective by (1).

(2)⇔(3). Let \( I \) be any finitely generated right ideal of \( R \) and \( (M_i, f_{ij})_\Lambda \) a direct system of right \( R \)-module with \( M_i \) \( f \)-injective for all \( i \in \Lambda \). Then the exactness of the sequence \( 0 \to I \to R \) induces the following commutative diagram:

\[
\begin{array}{ccccccccc}
\lim \text{Hom}(R, M_i) & \xrightarrow{\varphi} & \lim \text{Hom}(I, M_i) & \to & 0 \\
\downarrow & & \downarrow & & \\
\text{Hom}(R, \lim M_i) & \xrightarrow{\varphi_I} & \text{Hom}(I, \lim M_i) & \to & 0
\end{array}
\]

with \( \varphi \) being an isomorphism and \( \varphi_I \) being a monomorphism by Ref. [15, 24.9] (for \( I \) is finitely generated), where the first row is exact since \( M_i \in \mathcal{F}\text{-inj} \) for all \( i \in \Lambda \). So we have that \( R \) is right coherent if and only if \( I_R \) is a finitely presented right \( R \)-module if and only if \( \varphi_I \) is an isomorphism if and only if the bottom row is exact if and only if \( \lim M_i \) is \( f \)-injective.
Remark 3.4 Let m and n be fixed positive integers. Recall that a right R-module M is said to be $(m,n)$-injective\cite{ZhangXiao-xiang} if $\text{Ext}^1(P,M) = 0$ for any $(m,n)$-presented right R-module P; A ring R is called right $(m,n)$-coherent in case each n-generated submodule of the right R-module $R^m$ is finitely presented.

From the proof of Theorem 3.3, we may obtain the following general result:

$R$ is right $(m,n)$-coherent if and only if every direct limit of $(m,n)$-injective right R-modules is $(m,n)$-injective.

We conclude the paper with the following

Corollary 3.5 If $R$ is a left $(m,n)$-coherent ring, then $M^+$ has an $(m,n)$-injective cover for any right R-module M.

Proof By Ref.\cite{MaoLi-xin}, Corollary 3.3, \cite{EnochsE E}, Corollary 5.2.7 and Remark 3.4.

References