On the Crossing Numbers of $K_5 \times S_n$

LÜ Sheng Xiang\(^1\), HUANG Yuan Qiu\(^2\)

\(^1\)(1. Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China; 2. Department of Mathematics, Normal University of Hunan, Hunan 410081, China)

(E-mail: lsxx23@yahoo.com.cn; hyqq@public.cs.hn.cn)

Abstract By connecting the 5 vertices of $K_5$ to other $n$ vertices, we obtain a special family of graph denoted by $H_n$. This paper proves that the crossing number of $H_n$ is $Z(5, n) + 2n + \lfloor \frac{n^2}{2} \rfloor + 1$, and the crossing number of Cartesian products of $K_5$ with star $S_n$ is $Z(5, n) + 5n + \lfloor \frac{n^2}{2} \rfloor + 1$.

Keywords graph; drawing; crossing number; star; Cartesian products.

1. Introduction

For graph theory terminology not defined here we direct the reader to [1] and all the graphs are connected simple graphs. Let $G$ be a graph with vertex set $V_G$ and edge set $E_G$. For convenience, let $E_v$ be the set of all the edges incident to $v$ ($v \in V$) and for any edge set $E' \subseteq E$, let $\langle E' \rangle$ denote the edge-induced subgraph of $G$, and $G \{E' \}$ the subgraph of $G$ obtained by deleting the edges of edge set $E'$. Specially, $G \{e \}$ denotes the subgraph of $G$ obtained by deleting the edge $e$. If a vertex of graph $G$ has degree $k$, then we call it a $k$-vertex. Formally, the Cartesian product $G \times H$ of two graphs $G$ and $H$ has vertex set $V(G \times H) = V(G) \times V(H)$ and edge set

$$E(G \times H) = \{(u_i, v_j)(u_h, v_k) : u_i = u_h \text{ and } v_jv_k \in E(H) \text{ or } v_j = v_k \text{ and } u_iu_h \in E(G); u_i, u_h \in V(G), v_j, v_k \in V(H)\}.$$ 

If one of the graphs is $S_n$, the star $K_{1,n}$, a less formal description is helpful: $G \times S_n$ is obtained by $n + 1$ copies $G^0, G^1, \ldots, G^n$ of $G$ and by joining the vertices of the copies $G^i, i = 1, 2, \ldots, n$, to $G^0$ correspondingly.

A drawing is called good, if for all arcs in $A$, no two with a common endpoint meet, no two meet in more than one point, and no three have a common point. A crossing in a good drawing is a point of intersection of two arcs in $A$. A good drawing is said to be optimal if it minimizes the number of crossings. The crossing number $cr(G)$ of a graph $G$ is the number of crossings in
any optimal drawing of $G$ in the plane. Let $\phi$ be a good drawing of graph $G$. We denote by $cr_{\phi}(G)$ the number of crossings in $\phi$.

Gray and Johnson have proved that in general the problem to determine the crossing numbers of graphs is NP-complete\(^2\). At present only a few families of graphs with arbitrarily large crossing numbers for the plane are known. Guy and Zarankiewicz presented two important conjectures on the determination of the crossing numbers of graphs.

On the crossing numbers of complete graphs $K_n$, Guy conjectured:

$$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the maximum integer not greater than $x$. In 1993, Woodall proved that Guy’s conjecture is correct when $n \leq 10$ in \(^3\), but when $n \geq 11$, Guy’s conjecture is undetermined.

On the crossing numbers of complete bipartite graphs $K_{m,n}$, Zarankiewicz conjectured:

$$cr(K_{m,n}) = Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

In 1970, Kleitman proved that when $\min(m,n) \leq 6$, Zarankiewicz’s conjecture is true\(^4\); In \(^4\), Kleitman proved that Zarankiewicz’s conjecture is also true for $K_{7,7}$ and $K_{7,9}$. But for other $m,n$, Zarankiewicz’s conjecture is only an upper bound.

On the crossing numbers of complete tripartite graphs, Kouhei Asano determined the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$\(^5\).

On the crossing numbers of Cartesian product graphs, most are the Cartesian product of low order graphs with paths, cycles and stars\(^6\)\(^{13}\). Among them, Klešč determined the crossing numbers of $K_4 \times p_n$, $K_4 \times S_n$\(^8\) and $K_5 \times p_n$\(^9\); Beineke and Ringel determined the crossing numbers of $K_4 \times C_n$\(^10\). But the crossing numbers of $K_5 \times C_n$ and $K_5 \times S_n$ are undetermined.

In this paper, for the convenience of determining the crossing numbers of $K_5 \times S_n$, we first construct the following graph $H_n$:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Hn.png}
\caption{A good drawing of $H_n$}
\end{figure}

In the bipartite graph $K_{5,n}$ with bipartition $\{t_1, t_2, \ldots, t_n\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$, connecting the edges incident with any two vertices of set $\{v_1, v_2, v_3, v_4, v_5\}$ yields a new graph, denoted by $H_n$. Furthermore, $H_n$ can also be obtained by joining all the 5 vertices $\{v_1, v_2, v_3, v_4, v_5\}$ of $K_5$ to $n$ vertices $t_i$ ($i = 1, 2, \ldots, n$) (Figure 1). Let $T_i$ ($i = 1, 2, \ldots, n$) be the set of edges incident with vertex $t_i$ ($i = 1, 2, \ldots, n$) (Figure 2) and let $E_0$ be the set of edges of $K_5$. It is easy to
On the crossing numbers of $K_5 \times S_n$

obtain

$$E(H_n) = E_0 \cup \bigcup_{i=1}^{n} T_i.$$ 

The following theorems are our two main results:

**Theorem 1**  \( cr(H_n) = Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1. \)

**Theorem 2**  \( cr(K_5 \times S_n) = Z(5, n) + 5n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1. \)

2. The proofs of Theorems

In order to prove the Theorems, first we introduce some lemmas.

**Lemma 1**  If \( \phi \) is a good drawing of a graph \( G \), and \( E_1, E_2 \) and \( E_3 \) are three mutually disjoint edge subsets of \( G \), then we have

1) \( cr_\phi(E_1 \cup E_2) = cr_\phi(E_1) + cr_\phi(E_2); \)

2) \( cr_\phi(E_1 \cup E_2, E_3) = cr_\phi(E_1, E_3) + cr_\phi(E_2, E_3). \)

**Proof**  It is direct from the definitions. \( \Box \)

**Lemma 2**  If \( \phi \) is a good drawing of \( H_2 \) such that \( cr_\phi(T_1, T_2) = 0 \), then \( cr_\phi(E_0, T_1 \cup T_2) \geq 5. \)

![Figure 3 A good drawing \( \phi' \) of \( H \) when \( cr_\phi(T_1, T_2) = 0 \)](image)

**Proof**  Let \( H = \langle \{T^1 \cup T^2\} \rangle \) be the edge induced subgraph of \( H^2 \). Clearly, \( H \) is isomorphic to the complete bipartite graph \( K_{2,5} \) with vertex partition \( \{t^1, t^2\} \) and \( V(K_5) := \{v_1, v_2, v_3, v_4, v_5\} \). Since \( cr_\phi(T^1, T^2) = 0 \), the subdrawing \( \phi' \) of \( H \) induced by \( \phi \) must be isomorphic to Figure 3. From Figure 3, each edge of \( E'_0 := \{v_1v_3, v_1v_4, v_2v_4, v_2v_5, v_3v_5\} \) has at least one crossing with the edge set \( T_1 \cup T_2 \), then \( cr_\phi(E_0, T_1 \cup T_2) \geq 5. \) \( \Box \)

**Lemma 3**  \( cr(K_7 \setminus \{e\}) = 6, e \in E(K_7). \)

**Proof**  See the [9]. \( \Box \)

**Lemma 4**  \( cr(H_1) = 3, cr(H_2) = 6. \)

**Proof**  Since \( H_1 \) is isomorphic to \( K_6 \) and \( H_2 \) isomorphic to \( K_7 \setminus \{e\}, e \in E(K_7) \), we have
Further, as \( \langle 1, 2 \rangle \), we have

\[
T \left\lfloor \theta \text{ is even} \right) = cr(K_5) = 3; \ cr(H_2) = cr(K_7 \setminus \{e\}) = 6. \quad \square
\]

**Theorem 1** \( cr(H_n) = Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1. \)

**Proof** First construct a good drawing \( \theta \) of \( H_n \) as Figure 1. Clearly, in Figure 1, \( H_n \setminus E_0 \) is isomorphic to \( K_{5, n} \) and the induced subdrawing by \( \theta \) is an optimal drawing of \( K_{5, n} \), so \( cr(\theta(H_n \setminus E_0) = cr(K_{5, n}) = Z(5, n); cr(\theta(E_0, Ti) = 2 \) (when \( i \) is odd), \( cr(\theta(E_0, Ti) = 3 \) (when \( i \) is even); \( cr(\theta(E_0) = 1. \) Combining (1) and Lemma 1, we have

\[
cr(\theta(H_n) = cr(\theta(H_n \setminus E_0) + cr(\theta(E_0, \bigcup_{i=1}^{n} T_i) + cr(\theta(E_0))
= Z(5, n) + \sum_{i=1}^{n} cr(\theta(E_0, T_i)) + cr(\theta(E_0))
= Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\]

By the definition of crossing numbers, it is easy to know that \( cr(H_n) \leq cr(\theta(H_n)) = Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1, n \geq 1. \) In the following, we will prove that the theorem holds true by induction on \( n. \)

By Lemma 4, \( cr(H_1) = Z(5, 1) + 2 \times 1 + \lfloor \frac{1}{2} \rfloor + 1 = 3 \) and \( cr(H_2) = Z(5, 2) + 2 \times 2 + \lfloor \frac{2}{2} \rfloor + 1 = 6. \) So the Theorem holds when \( n = 1, 2. \) Now assume that \( n \geq 3 \) and when \( \ell < n, \) \( cr(H_\ell) = Z(5, \ell) + 2\ell + \lfloor \frac{\ell}{2} \rfloor + 1. \) Let \( \phi \) be a good drawing of \( H_n. \) Then we only need to prove that \( cr(\phi(H_n) \geq Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1. \) We divide the problem into the following different cases:

**Case 1** There exist two vertices \( t_i \) and \( t_j (1 \leq i, j \leq n; i \neq j) \) such that \( cr(\phi(T_i, T_j) = 0. \)

Without loss of generality, assume that \( cr(\phi(T_n, T_{n-1}) = 0. \) When \( 1 \leq i \leq n - 2, \) as \( (T_n \cup T_{n-1} \cup T_i) \) is isomorphic to complete bipartite graph \( K_{3, 5} \) with \( cr(\phi(T_n, T_{n-1}) = 0, \) we have

\[
cr(\phi(T_n \cup T_{n-1}, T_i) = cr(\phi(K_{3, 5}) - cr(\phi(T_n \cup T_{n-1}) - cr(\phi(T_i) \geq 4 - 0 - 0 = 4. \quad (2)
\]

Furthermore, as \( (E_0 \cup (\bigcup_{i=1}^{n-2} T_i)) \) is isomorphic to \( H_{n-2}, \) combining Lemmas 1, 2 and Equations (1), (2), we have

\[
cr(\phi(H_n) = cr(\phi(E_0 \cup T_n \cup T_{n-1} \cup \bigcup_{i=1}^{n-2} T_i)
= cr(\phi(T_n \cup T_{n-1} \cup \bigcup_{i=1}^{n-2} T_i) + cr(\phi(T_n \cup T_{n-1} \cup E_0) + cr(\phi(E_0 \cup \bigcup_{i=1}^{n-2} T_i))
= \sum_{i=1}^{n-2} cr(\phi(T_n \cup T_{n-1} \cup T_i) + cr(\phi(T_n \cup T_{n-1} \cup E_0) + cr(\phi(E_0 \cup \bigcup_{i=1}^{n-2} T_i))
\geq 4(n - 2) + 5 + Z(5, n - 2) + 2(n - 2) + \lfloor \frac{n - 2}{2} \rfloor + 1
= Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.
\]

**Case 2** For every \( 1 \leq i < j \leq n, \) there holds \( cr(\phi(T_i, T_j) \geq 1. \)

**Subcase 2.1** There exists a vertex \( t_i (1 \leq i \leq n) \) such that \( cr(\phi(T_i, E_0)) = 0. \)
Suppose \( \text{cr}_\phi(H_n) < Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 \). Without loss of generality, let \( \text{cr}_\phi(T_n, E_0) = 0 \). Then consider the subdrawing \( \phi_1 \) of \( (T_n \cup E_0) \) induced by \( \phi \): as \( \text{cr}_\phi(T_n, E_0) = 0 \), there is a disk \( C \) such that the vertices of \( K_5 \) are all located on the boundary of \( C \), and the edges of \( K_5 \) are all located in the inner of \( C \). Furthermore, as \( \phi \) is a good drawing and the edges of \( K_5 \) can be presented by straight lines, vertex \( t_1 \) and the edges incident with \( t_1 \) are all located on the outside of \( C \), see Figure 4. In Figure 4, we divide the regions of \( \phi_1 \) into 2 classes:

1) On \( \phi \), when vertices \( t_i \) (\( 1 \leq i \leq n - 1 \)) are located in the region marked with 1, then \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 4 \), and we can find that only if \( \text{cr}_\phi(T_i, E_0) = 0 \), there is \( \text{cr}_\phi(T_i, E_0 \cup T_n) = 4 \); if \( \text{cr}_\phi(T_i, E_0) > 0 \), then \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5 \).

2) On \( \phi \), when vertices \( t_i \) (\( 1 \leq i \leq n - 1 \)) are located in the region marked with 2, 3, 4, then \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5 \).

Let \( A_1 := \{t_i | \text{cr}_\phi(T_i, E_0) = 0, 1 \leq i \leq n - 1 \} \).

Let \( A_2 := \{t_i | \text{cr}_\phi(T_i, E_0) > 0, 1 \leq i \leq n - 1 \} \).

So \( A_1 \) presents set of some of the vertices located in region 1, and for any \( t_i \in A_1 \), \( \text{cr}_\phi(T_i, T_n) \geq 4 \); also, \( A_1 \cup A_2 = \{t_i, 1 \leq i \leq n - 1 \} \). As \( \langle E_0 \cup \bigcup_{i=1}^{n-1} T_i \rangle \) is isomorphic to \( H_{n-1} \), by Lemma 1, we get

\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)
= \text{cr}_\phi(T_n, \bigcup_{i=1}^{n-1} T_i) + \text{cr}_\phi(T_n, E_0) + \sum_{i=1}^{n-1} \text{cr}_\phi(E_0 \cup T_i)
= \text{cr}_\phi(T_n, \bigcup_{t_i \in A_1} T_i) + \text{cr}_\phi(T_n, \bigcup_{t_i \in A_2} T_i) + \text{cr}_\phi(T_n, E_0) + \sum_{i=1}^{n-1} \text{cr}_\phi(E_0 \cup T_i)
\geq 2|A_1| + |A_2| + 0 + Z(5, n - 1) + 2(n - 1) + \left\lfloor \frac{n-1}{2} \right\rfloor + 1
\]
= Z(5, n - 1) + 2(n - 1) + \left\lceil \frac{n - 1}{2} \right\rceil + 1 + (n - 1) + 3|A_1| \\
= Z(5, n - 1) + 3(n - 1) + \left\lceil \frac{n - 1}{2} \right\rceil + 1 + 3|A_1|.

But at the beginning, we assume that \( cr_\phi(H_n) < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 \). By a simple calculation, we have

\[
|A_1| < \frac{1}{3}(Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 - (Z(5, n - 1) + 3(n - 1) + \left\lceil \frac{n - 1}{2} \right\rceil + 1)) \leq \frac{n + 1}{3}. \tag{3}
\]

Also \( \langle \bigcup_{i=1}^{n-1} T_i \rangle \) is isomorphic to \( K_{5,n-1} \). By Lemma 1 and Equations (1) and (3), we get

\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)
= \text{cr}_\phi(T_n \cup E_0, \bigcup_{i=1}^{n-1} T_i) + \text{cr}_\phi(E_0 \cup T_n) + \text{cr}_\phi\left(\bigcup_{i=1}^{n-1} T_i\right)
= \text{cr}_\phi(T_n \cup E_0, \bigcup_{t_i \in A_1} T_i) + \text{cr}_\phi(T_n \cup E_0, \bigcup_{t_i \in A_2} T_i) + \text{cr}_\phi(E_0 \cup T_n) + \text{cr}_\phi\left(\bigcup_{i=1}^{n-1} T_i\right)
= \sum_{t_i \in A_1} \text{cr}_\phi(T_n \cup E_0, T_i) + \sum_{t_i \in A_2} \text{cr}_\phi(T_n \cup E_0, T_i) + \text{cr}_\phi(E_0 \cup T_n) + \text{cr}_\phi\left(\bigcup_{i=1}^{n-1} T_i\right)
\geq 4|A_1| + 5|A_2| + 5 + Z(5, n - 1)
= Z(5, n - 1) + 4|A_1| + 5(n - 1 - |A_1|) + 5
= Z(5, n - 1) + 5(n - 1) - |A_1| + 5
> Z(5, n - 1) + 5n - \frac{n + 1}{3}
\geq Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1.
\]

**Subcase 2.2** For any \( t_i \) (\( 1 \leq i \leq n \)), \( \text{cr}_\phi(T_i, E_0) \geq 1 \), and there exists a vertex \( t_i \) (\( 1 \leq i \leq n \)) such that \( \text{cr}_\phi(T_i, E_0) = 1 \).

Without loss of generality, suppose \( \text{cr}_\phi(T_n, E_0) = 1 \). Let \( \phi_2 \) be the subdrawing of \( (T_n \cup E_0) \) induced by \( \phi \). Now we will explain there is only one good drawing \( \phi_2 \) of \( (T_n \cup E_0) \): First, suppose the edge \( t_nv_1 \) of \( T_n \) and the edge \( v_2v_4 \) of \( K_5 \) cross each other. We can suppose the vertices \( t_n, v_1, v_3, v_4 \) are located on the plane \( R^2 \) as in Figure 5(a), and the other three vertices of \( K_5 \) can be located arbitrarily around the vertices \( t_n, v_1, v_3, v_4 \). But as there is no crossing on the other 4 edges which are incident with vertex \( t_n \), and the edges incident with \( v_1 \) of \( K_5 \) cannot cross the edges of \( T^n \), these edges can be drawn as shown in Figure 5(a) and the edge \( v_2v_3 \) cannot be drawn as the dotted line as shown in the Figure 5(a) or \( \text{cr}_\phi(T_n, E_0) \geq 2 \) does not hold. As for the other edges we can draw them analogously, and the only difference is that they are connected by straight lines or by arcs, but they are isomorphic to each other. So the subdrawing \( \phi_2 \) of
On the crossing numbers of $K_5 \times S_n$

The crossing number of $\langle T_n \cup E_0 \rangle$ must be isomorphic to Figure 5(b).

![Figure 5](image)

Figure 5 A good drawing $\phi_2$ of $\langle T_n \cup E_0 \rangle$ when $cr_\phi(T_n, E_0) = 1$

In Figure 5(b), if $t_i$ $(1 \leq i \leq n-1)$ is located in the region marked with 1, then $cr_\phi(T_i, E_0) \geq 4$, and with hypothesis $cr_\phi(T_i, T_j) \geq 1$ $(1 \leq i < j \leq n)$, $cr_\phi(T_i, E_0 \cup T_n) \geq 5$; if $t_i$ $(1 \leq i \leq n-1)$ is located in the other region, then $cr_\phi(T_i, E_0 \cup T_n) \geq 5$. So no matter which region the vertex $t_i$ $(1 \leq i \leq n-1)$ is located in, we always have $cr_\phi(T_i, E_0 \cup T_n) \geq 5$. Also, $cr_\phi(E_0 \cup T_n) = 4$ and $\bigcup_{i=1}^{n-1} T_i$ is isomorphic to $K_{5,n-1}$. By Lemma 1, we get

$$cr_\phi(H_n) = cr_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)$$

$$= cr_\phi(E_0 \cup T_n, \bigcup_{i=1}^{n-1} T_i) + cr_\phi(E_0 \cup T_n) + cr_\phi(\bigcup_{i=1}^{n-1} T_i)$$

$$= \sum_{i=1}^{n-1} cr_\phi(E_0 \cup T_n, T_i) + cr_\phi(E_0 \cup T_n) + cr_\phi(\bigcup_{i=1}^{n-1} T_i)$$

$$\geq 5(n-1) + 4 + Z(5,n-1)$$

$$\geq Z(5,n) + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$  

**Subcase 2.3** For any $t_i$ $(1 \leq i \leq n)$, $cr_\phi(T_i, E_0) \geq 2$, and there exists a vertex $t_i$ $(1 \leq i \leq n)$ such that $cr_\phi(T_i, E_0) = 2$.

Without loss of generality, suppose $cr_\phi(T_n, E_0) = 2$. Let $\phi_3$ of $\langle T_n \cup E_0 \rangle$ be induced by $\phi$. By using the same method as in Subcase 2.2, we divide the subdrawing $\phi_3$ of $\langle T_n \cup E_0 \rangle$ into 3 different cases:

1) Two edges of $T_n$ cross with one edge of $E_0$. In this case, there is only one drawing of $\phi_3$, see Figure 6;

2) One edge of $T_n$ crosses with two edges of $E_0$. In this case, there are three different drawings of $\phi_3$, see Figure 7, Figure 8 and Figure 9; Figure 7 presents the case that one edge of $T_n$ crosses with two adjacent edges of $E_0$; Figure 8 presents the case that one edge of $T_n$ crosses with two unadjacent edges $e_1$ and $e_2$ of $E_0$, where $e_1$ and $e_2$ do not cross each other; Figure 9
presents the case that one edge of $T_n$ crosses with two unadjacent edges $e_1$ and $e_2$ of $E_0$, where $e_1$ and $e_2$ cross each other.

![Figure 6](image1)

![Figure 7](image2)

![Figure 8](image3)

![Figure 9](image4)

![Figure 10](image5)

Five cases of good drawing $\phi_3$ of $\langle T_n \cup E_0 \rangle$ when $cr_{\phi}(T_n, E_0) = 2$

3) Two edges of $T_n$ cross with two edges of $E_0$. In this case, there is only one drawing of $\phi_3$, see Figure 10.

In the following we will discuss the different drawings of $\phi_3$.

1) The subdrawing $\phi_3$ of $\langle T_n \cup E_0 \rangle$ is isomorphic to Figure 6. By Figure 6, we know, if $t_i$ ($1 \leq i \leq n - 1$) is located in the region marked with 1, then $cr_{\phi}(T_i, E_0) \geq 4$, and by hypothesis that $cr_{\phi}(T_i, T_j) \geq 1$ ($1 \leq i < j \leq n$), $cr_{\phi}(T_i, E_0 \cup T_n) \geq 5$; Also if $t_i$ ($1 \leq i \leq n - 1$) is located in the other regions, it is easy to obtain that $cr_{\phi}(T_i, E_0 \cup T_n) \geq 5$. So no matter which region of Figure 6 the vertex $t_i$ ($1 \leq i \leq n - 1$) is located in, there always holds $cr_{\phi}(T_i, E_0 \cup T_n) \geq 5$. As $cr_{\phi}(E_0 \cup T_n) = 5$ and $\bigcup_{i=1}^{n-1} T_i$ is isomorphic to complete bipartite graph $K_{5,n-1}$, we have by Lemma 1

$$cr_{\phi}(H_n) = cr_{\phi}(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)$$
2) The subdrawing \( \phi \) of \( \langle T_n \cup E_0 \rangle \) is isomorphic to Figure 7. First assume that \( \text{cr}_\phi(H_n) < Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 \). By Figure 7, if \( t_i \) \((1 \leq i \leq n - 1)\) is located in the region marked with 1, then we can also prove \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5 \) with the hypothesis that \( \text{cr}_\phi(T_i, E_0) \geq 2 \); if \( t_i \) \((1 \leq i \leq n - 1)\) is located in the region marked with 2, then \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 4 \), and only if \( \text{cr}_\phi(T_i, E_0) = 2 \) and \( \text{cr}_\phi(T_i, T_n) = 2 \), there holds \( \text{cr}_\phi(T_i, E_0 \cup T_n) = 4 \), see \( t_1 \) in Figure 7; if \( t_i \) \((1 \leq i \leq n - 1)\) is located in the region marked with 3, then \( \text{cr}_\phi(T_i, E_0) \geq 4 \). By the hypothesis that \( \text{cr}_\phi(T_i, T_j) \geq 1 \) \((1 \leq i, j \leq n; i \neq j)\), we have \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5 \); if \( t_i \) \((1 \leq i \leq n - 1)\) is located in the other regions, it is easy to know that \( \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5 \).

Let \( B_1 := \{ t_i | \text{cr}_\phi(T_i, E_0 \cup T_n) = 4, 1 \leq i \leq n - 1 \} \) and \( B_2 := \{ t_i | \text{cr}_\phi(T_i, E_0 \cup T_n) \geq 5, 1 \leq i \leq n - 1 \} \). From the definition above, we know that if \( t_i \in B_1 \), then \( \text{cr}_\phi(T_i, E_0) = 2 \), \( \text{cr}_\phi(T_i, T_n) = 2 \) and \( B_1 \cup B_2 = \{ t_i, 1 \leq i \leq n - 1 \} \). Also \( \text{cr}_\phi(E_0 \cup T_n) = 5 \), and \( \langle \bigcup_{i=1}^{n-1} T_i \rangle \) is isomorphic to complete bipartite graph \( K_{5, n-1} \). So by Lemma 1, we have

\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i) = \text{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in B_1} T_i) + \text{cr}_\phi(E_0 \cup T_n, \bigcup_{t_i \in B_2} T_i) + \text{cr}_\phi(E_0 \cup T_n) + \text{cr}_\phi(\bigcup_{i=1}^{n-1} T_i) \geq 4|B_1| + 5|B_2| + 5 + Z(5, n - 1) = 5(n - 1) - |B_1| + 5 + Z(5, n - 1).
\]

By the assumption at the beginning \( \text{cr}_\phi(H_n) < Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 \) and by a simple calculation, we have

\[ |B_1| > Z(5, n - 1) + 5(n - 1) - 5 - (Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 1) \geq \lfloor \frac{n}{2} \rfloor + 2. \tag{4} \]

So \( |B_1| \neq 0 \). Let \( t_1 \in B_1 \). By the analogous discussion, for any \( t_i \) \((2 \leq i \leq n)\), \( \text{cr}_\phi(T_i, E_0 \cup T_1) \geq 4 \). If \( t_i \in B_1 \) \((2 \leq i \leq n - 1)\), then we have by observation, \( \text{cr}_\phi(T_i, E_0 \cup T_1) \geq 6 \).

Let \( B_3 := \{ t_i | \text{cr}_\phi(T_i, E_0 \cup T_1) = 4, 2 \leq i \leq n \} \).

Let \( B_4 := \{ t_i | \text{cr}_\phi(T_i, E_0 \cup T_1) \geq 5, 2 \leq i \leq n, \} \).

Obviously, \( B_1 \cap B_3 = 0 \), \( B_3 \cup B_4 = \{ t_i, 2 \leq i \leq n \} \) and \( B_1 \cup B_3 \subseteq \{ t_i, 1 \leq i \leq n \} \), so

\[ |B_1 \cup B_3| \leq n. \tag{5} \]
By discussing analogously to Equation (4), we also have

$$|B_3| \geq \left\lceil \frac{n}{2} \right\rceil + 2. \tag{6}$$

Combining (4) and (6), we get

$$|B_1 \cup B_3| = |B_1| + |B_3| \geq n + 3. \tag{7}$$

This contradicts with Equation (5). So $cr_\phi(H_n) \geq Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1$.

3) The subdrawing $\phi_3$ of $(T_n \cup E_0)$ is isomorphic to Figure 8. By Figure 8, if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 1, then $cr_\phi(T_i, E_0 \cup T_n) \geq 4$, and only if $cr_\phi(T_i, E_0) = 2$ and $cr_\phi(T_i, T_n) = 2$, the equation $cr_\phi(T_i, E_0 \cup T_n) = 4$ holds, just like the $t_1$ in Figure 8; if $t_i \ (1 \leq i \leq n-1)$ is located in the other regions, then $cr_\phi(T_i, E_0 \cup T_n) \geq 5$. So by using an analogous method as in Figure 7, one can also obtain that $cr_\phi(H_n) \geq Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1$. As for Figure 9, the discussion is the same as Figure 8.

4) The subdrawing $\phi_3$ of $(T_n \cup E_0)$ is isomorphic to Figure 10. First assume that

$$cr_\phi(H_n) < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1. \tag{8}$$

In Figure 10, if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 5, then $cr_\phi(T_i, E_0 \cup T_n) \geq 5$; if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 4, then $cr_\phi(T_i, E_0 \cup T_n) \geq 6$; if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 3, then $cr_\phi(T_i, E_0 \cup T_n) \geq 4$ and $cr_\phi(T_i, E_0) \geq 3$; if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 2, then $cr_\phi(T_i, E_0) \geq 3$, and with the hypothesis $cr_\phi(T_i, T_n) \geq 1$, then $cr_\phi(T_i, E_0 \cup T_n) \geq 4$; if $t_i \ (1 \leq i \leq n-1)$ is located in the region marked with 1, then $cr_\phi(T_i, E_0 \cup T_n) \geq 4$, and if $cr_\phi(T_i, E_0) = 2$, we must have $cr_\phi(T_i, T_n) = 2$ and $cr_\phi(T_i, E_0 \cup T_n) = 4$.

Let $\Omega := \{t_i, 1 \leq i \leq n-1\}$.

Let $C_1$ and $C_2$ denote the sets of the vertices $t_i, 1 \leq i \leq n-1$, located in region 4 and in region 5, respectively.

Let $D = \Omega \setminus (C_1 \cup C_2)$.

Let $C_3 := \{t_i | cr_\phi(T_i, E_0 \cup T_n) = 4, cr_\phi(T_i, T_n) = 2, cr_\phi(T_i, E_0) = 2, 1 \leq i \leq n-1\}$. Obviously, $C_3$ denotes the set of the vertices located in region 1 and has exact two crossings with edge set $E_0$ and $T_n$ respectively, and the lines connecting the vertices of $C_3$ and the 5 vertices of $K_5$ are all like the dotted line as shown in Figure 10. But the vertex which is located in region 1 and does not belong to $C_3$ has at least 3 crossings with the edges set $E_0$.

By the observation, if $t_i \in C_1, t_j \in C_2, t_k, t_l \in C_3$, then

$$cr_\phi(T_i, E_0 \cup T_k) \geq 4, \tag{9}$$
$$cr_\phi(T_j, E_0 \cup T_k) \geq 5, \tag{10}$$
$$cr_\phi(T_l, E_0 \cup T_k) \geq 6. \tag{11}$$

First we assume that $C_3 = \emptyset$. As $C_1 \cap C_2 = \emptyset$, $(\bigcup_{i=1}^{n-1} T_i)$ is isomorphic to complete bipartite
graph $K_{5,n-1}$ and $cr_\phi(E_0 \cup T_n) = 3$. So by Equation (1) and Lemma 1, we obtain

$$cr_\phi(H_n) = cr_\phi(E_0 \cup T_n \cup \bigcup_{i=1}^{n-1} T_i)$$

$$= cr_\phi(E_0 \cup T_n, \bigcup_{t_i \in (C_1 \cup C_2)} T_i) + \sum_{t_i \in D} cr_\phi(E_0 \cup T_n, T_i) +$$

$$cr_\phi(E_0 \cup T_n) + cr_\phi(\bigcup_{i=1}^{n-1} T_i)$$

$$\geq 5|C_1 \cup C_2| + 4|D| + 3 + Z(5,n-1)$$

$$= 5(|C_1| + |C_2|) + 4(n-1) - |C_1| - |C_2| + 3 + Z(5,n-1)$$

$$= 4(n-1) + ((|C_1| + |C_2|) + 3 + Z(5,n-1)).$$

Also with the hypothesis that $cr_\phi(H_n) < Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$, we have

$$|C_1| + |C_2| \leq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1 - (Z(5,n-1) + 4(n-1) + 3) - 1 \leq \lfloor \frac{n}{2} \rfloor - 1. \quad (12)$$

Under the assumption $C_3 = \emptyset$, if $t_i \in D$, then $cr_\phi(T_i, E_0) \geq 3$. Combining (1), (12) and Lemma 1, we get

$$cr_\phi(H_n) = cr_\phi(E_0 \cup \bigcup_{i=1}^{n} T_i)$$

$$= cr_\phi(E_0, \bigcup_{t_i \in (C_1 \cup C_2)} T_i) + \sum_{t_i \in D} cr_\phi(E_0, T_i) +$$

$$cr_\phi(E_0, T_n) + cr_\phi(E_0) + cr_\phi(\bigcup_{i=1}^{n} T_i)$$

$$\geq 2(|C_1| + |C_2|) + 3(n-1) - |C_1| - |C_2| + 2 + 1 + Z(5,n)$$

$$= 3(n-1) - (|C_1| + |C_2|) + 3 + Z(5,n)$$

$$\geq 3(n-1) - \lfloor \frac{n}{2} \rfloor - 1 + 3 + Z(5,n)$$

$$= Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$$

$$\geq Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1.$$
By the assumption (8), we get:

When \( n \) is odd

\[
2|C_1| + |C_2| < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 - (Z(5, n - 1) + 4(n - 1) + 3) = \left\lceil \frac{n}{2} \right\rceil.
\]  

(13)

When \( n \) is even

\[
2|C_1| + |C_2| < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 - (Z(5, n - 1) + 4(n - 1) + 3) = \left\lceil \frac{n}{2} \right\rceil - 2.
\]  

(14)

Furthermore, by the definition of set \( C_2, C_3, D' \) and Equations (9), (10) and (11), we also have

\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup T_1 \cup \bigcup_{i=2}^{n} T_i)
= \text{cr}_\phi(E_0 \cup T_1, \bigcup_{i \in C_2}^{n} T_i) + \text{cr}_\phi(E_0 \cup T_1, \bigcup_{i \in C_3}^{n} T_i) + \text{cr}_\phi(E_0 \cup T_1, \bigcup_{i \in D'}^{n} T_i)
\geq 6|C_3'| + 5|C_2| + 4|D'| + 3 + Z(5, n - 1)
= 6|C_3'| + 5|C_2| + 4(n - 1 - |C_3'| - |C_2|) + 3 + Z(5, n - 1)
= 4(n - 1) + 2|C_3'| + |C_2| + 3 + Z(5, n - 1).
\]

By the assumption (8), there also holds the following:

When \( n \) is odd

\[
2|C_3'| + |C_2| < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 - (Z(5, n - 1) + 4(n - 1) + 3) = \left\lceil \frac{n}{2} \right\rceil.
\]  

(15)

When \( n \) is even

\[
2|C_3'| + |C_2| < Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 - (Z(5, n - 1) + 4(n - 1) + 3) = \left\lceil \frac{n}{2} \right\rceil - 2.
\]  

(16)

So when \( n \) is odd, by adding (13) to (15), and then dividing 2, we get

\[
|C_1| + |C_2| + |C_3'| \leq \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lceil \frac{n}{2} \right\rceil - 2.
\]  

(17)

When \( n \) is even, by adding (14) to (16) and then dividing 2, we get

\[
|C_1| + |C_2| + |C_3'| \leq \left\lceil \frac{n}{2} \right\rceil - 3 = \left\lceil \frac{n}{2} \right\rceil - 3.
\]  

(18)

Let \( \Lambda = (C_1 \cup C_2 \cup C_3 \cup \{t_1, t_2\}) \) and \( \Gamma = \Omega \setminus (C_1 \cup C_2 \cup C_3) \). By the definition of the sets above, it is easy to know: if \( t_i \in \Lambda \), then \( \text{cr}_\phi(E_0, t_i) \geq 2 \); if \( t_i \in \Gamma \), then \( \text{cr}_\phi(E_0, t_i) \geq 3 \). So by Lemma
1, we get
\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup \bigcup_{i=1}^n T_i)
\]
\[
= \text{cr}_\phi(E_0, \bigcup_{t_i \in \Lambda} T_i) + \text{cr}_\phi(E_0, \bigcup_{t_i \in \Gamma} T_i) + \text{cr}_\phi(E_0) + \text{cr}_\phi(\bigcup_{i=1}^n T_i)
\]
\[
\geq 2(|C_1| + |C_2| + |C_3'| + 2) + 3(n - 2 - |C_1| - |C_2| - |C_3'|) + 1 + Z(5, n)
\]
\[
= 3n - (|C_1| + |C_2| + |C_3'|) + Z(5, n) - 1.
\]
Combining the assumption (8), we also have
\[
|C_1| + |C_2| + |C_3'| > Z(5, n) + 3n - 1 - (Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1) = \left\lceil \frac{n}{2} \right\rceil - 2. \quad (19)
\]
But they are contradicting with Equations (17) and (18). So we have \( \text{cr}_\phi(H_n) \geq Z(5, n) + 2n + \left\lceil \frac{n}{2} \right\rceil + 1 \).

**Subcase 2.4** For any \( t_i \) (1 \( \leq i \leq n \)), \( \text{cr}_\phi(T_i, E_0) \geq 3 \).

By Lemma 1 and Equation (1), it is easy to get
\[
\text{cr}_\phi(H_n) = \text{cr}_\phi(E_0 \cup \bigcup_{i=1}^n T_i) = \text{cr}_\phi(E_0, \bigcup_{i=1} T_i) + \text{cr}_\phi(E_0) + \text{cr}_\phi(\bigcup_{i=1}^n T_i)
\]
\[
\geq 3n + 1 + Z(5, n).
\]
Now the proof is completed. \( \square \)

Let \( H \) be a graph isomorphic to \( K_5 \). Consider a graph \( G_H \) obtained by joining all vertices of \( H \) to five vertices of a 3-edge connected graph \( G \) such that every vertex of \( H \) will only be adjacent to exactly one vertex of \( G \). Let \( G_H^* \) be the graph obtained by contracting all the edges of \( H \) to a vertex \( h \).

**Lemma 5** \( \text{cr}(G_H^*) \leq \text{cr}(G_H) - 3 \).

**Proof** Let \( \psi \) be the optimal drawing of \( G_H \). By joining all the 5 vertices of \( H \) to a vertex \( z \), we will obtain a graph isomorphic to \( K_6 \). As \( \text{cr}(K_6) \geq 3 \), in \( \psi \) there are at least 3 crossings on the edges of \( H \). Let edge set \( E_2 := \{v_2v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_4v_5\} \), see Figure 11 and Figure 12. Then we divide into 2 cases by number of crossings on the edge set \( E_2 \):

**Case 1** On \( \psi \), there are at least 3 crossings on \( E_2 \). As \( H \) includes a subgraph \( S_4 \) with the vertex \( v_1 \) as a 4-vertex, \( G_H \setminus E_2 \) is isomorphic to \( G_H^* \), and \( \text{cr}(G_H^*) = \text{cr}(G_H \setminus E_2) \leq \text{cr}_\psi(G_H \setminus E_2) \leq \text{cr}_\psi(G_H) - 3 = \text{cr}(G_H) - 3 \), see Figure 11.

**Case 2** On \( \psi \), there are at most 2 crossings on \( E_2 \). Then by the parity of crossing numbers\(^{[14]}\), \( \text{cr}_\psi(H) = 1 \), and there is only one subdrawing of graph \( H \) under \( \psi \) (see Figure 12). So there are at most one crossing on the edge set \( \{v_2v_3, v_3v_4, v_4v_5, v_5v_2\} \) (see Figure 12), say, on the edge \( v_4v_5 \). On the edges incident with vertex \( v_1 \), contracting along the edge, which has the minimum crossings, to the vertex \( h \) as shown in Figure 12 will also decrease at least 3 crossings. In Figure
12, we assume the edge $v_1v_3$ has the minimum crossing.

Figure 11 $G_H \backslash E_2$ isomorphic to $G_H^*$ when there are at least 3 crossings on $E_2$

Figure 12 The contraction when there are at most 2 crossings on $E_2$

Figure 13 An optimal drawing of $K_5 \times S_n$

**Theorem 2** $cr(K_5 \times S_n) = Z(5,n) + 5n + \lceil \frac{n}{2} \rceil + 1, n \geq 1.$

**Proof** In Figure 13, by contracting each copy $K_5^i (i = 1, 2, \ldots, n)$ to a vertex $t_i$, we will obtain a graph which is isomorphic to $H_n$, and the drawing is just an optimal drawing of $H_n$. Through the contracting, each copy $K_5^i (i = 1, 2, \ldots, n)$ decreases exactly 3 crossings, so $cr(K_5 \times S_n) \leq Z(5,n) + 5n + \lceil \frac{n}{2} \rceil + 1$. In the following, we will prove the opposite inequality holds. Let $\varphi$ be
On the crossing numbers of $K_5 \times S_n$

the optimal drawing of $K_5 \times S_n$. On $\varphi$, contracting each $K^5_i$ to a vertex $t_i$ ($i = 1, 2, \ldots, n$) yields a graph isomorphic to $H_n$. According to Theorem 1 and by using Lemma 5 repeatedly, we have

$$\text{cr}(K_5 \times S_n) = \text{cr}_\varphi(K_5 \times S_n) \geq \text{cr}(H_n) + 3n$$

$$= Z(5, n) + 2n + \left\lfloor \frac{n}{2} \right\rfloor + 1 + 3n = Z(5, n) + 5n + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

The proof is completed.

References