A Note on Adjacent-Vertex-Distinguishing Total Chromatic Numbers for $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$

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Abstract Let $G$ be a simple graph. Let $f$ be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$. Let $C_f(v) = \{f(v)\} \cup \{f(wv)|w \in V(G), vw \in E(G)\}$ for every $v \in V(G)$. If $f$ is a $k$-proper-total-coloring, and for $u, v \in V(G), uv \in E(G)$, we have $C_f(u) \neq C_f(v)$, then $f$ is called a $k$-adjacent-vertex-distinguishing total coloring ($k$-AVDT for short). Let $\chi_{at}(G) = \min\{k|G$ have a $k$-adjacent-vertex-distinguishing total coloring}. Then $\chi_{at}(G)$ is called the adjacent-vertex-distinguishing total chromatic number (AVDT number for short). The AVDT numbers for $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$ are obtained in this paper.

Keywords total coloring; adjacent-vertex-distinguishing total coloring; adjacent-vertex-distinguishing total chromatic number.

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1. Introduction

The graphs considered in this paper are connected, finite, undirected and simple graphs. In [1, 2, 3, 5] the vertex-distinguishing proper edge coloring (i.e. strong coloring), proper edge coloring of a graph in which no two of its vertices is incident to edges colored with the same set of colors, was introduced and investigated. In [7] the adjacent strong edge coloring (i.e. adjacent-vertex-distinguishing proper edge coloring), proper edge coloring of a graph $G$ in which no two adjacent vertices of $G$ is incident to edges colored with the same set of colors, was introduced and studied by ZHANG Zhongfu et al. These concepts can be generalized. The adjacent-vertex-distinguishing total coloring was introduced in [8]. A $k$-proper-total-coloring $f$ of a graph $G$ is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \ldots, k\}$ such that the following 3 conditions are valid:

1) $\forall u, v \in V(G)$, if $uv \in E(G)$, then $f(u) \neq f(v)$;
2) $\forall e_1, e_2 \in E(G), e_1 \neq e_2$, if $e_1, e_2$ have a common end vertex, then $f(e_1) \neq f(e_2)$;
3) $\forall u \in V(G), e \in E(G)$, if $u$ is an end vertex of $e$, then $f(u) \neq f(e)$.

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Conjecture 1 \textsuperscript{[8]} For every graph $G$ with order at least 2, we have $\chi_{at}(G) \leq \Delta(G) + 3$.

Note that for complete graph $G$ with order odd and at least 3, we have $\chi_{at}(G) = \Delta(G) + 3$.

Let $G$ and $H$ be graphs. Suppose that $V(G) = \{u_1, u_2, \ldots, u_m\}$, $V(H) = \{v_1, v_2, \ldots, v_n\}$. The Cartesian product of $G$ and $H$, denoted by $G \times H$, is defined as follows: $V(G \times H) = \{w_{ij} | i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, $E(G \times H) = \{w_{ij}w_{rs} | i = r, v_jv_s \in E(H) \text{ or } j = s, v_iv_r \in E(G)\}$. Let $P_n$ be a path with $n$ vertices and $C_n$ be a cycle with $n$ vertices. The adjacent-vertex-distinguishing total coloring on $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$ are studied and the corresponding chromatic numbers are obtained by constructing 4, 5, 6-AVDTC in this paper. Theorems 1, 2 and 3 in this paper will indicate that Conjecture 1 is valid for $P_m \times P_n$, $P_m \times C_n$ and $C_m \times C_n$. For the graph-theoretic terminology the reader is referred to \cite{4, 6}. The following lemma is obvious.

Lemma 1 If arbitrary two distinct vertices of maximum degree in $G$ are not adjacent, then $\chi_{at}(G) \geq \Delta(G) + 1$; If $G$ has two distinct vertices of maximum degree which are adjacent, then $\chi_{at}(G) \geq \Delta(G) + 2$.

2. The AVDTC number for $P_m \times P_n$

Theorem 1 Let $2 \leq m \leq n$. Then $\chi_{at}(P_m \times P_n) = \begin{cases} 
4, & m = n = 2; \\
5, & m = 2, n \geq 3 \text{ or } m = n = 3; \\
6, & m = 3, n \geq 4 \text{ or } n \geq m \geq 4.
\end{cases}$

Proof Assume that $P_m = u_1u_2 \cdots u_m$, $P_n = v_1v_2 \cdots v_n$, and $V(P_m \times P_n) = \{w_{ij} | i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, $E(P_m \times P_n) = \{w_{ij}w_{rs} | i = r, v_jv_s \in E(P_n) \text{ or } j = s, v_iv_r \in E(P_m)\}$. There are 4 cases to be considered.

Case 1 $m = n = 2$.

In this case, $P_2 \times P_2 = C_4$. Obviously, we have that $\chi_{at}(P_2 \times P_2) = \chi_{at}(C_4) = 4$.

Case 2 $m = 2, n \geq 3$.

In this case, there exist two adjacent vertices of degree 3. So $\chi_{at}(P_2 \times P_n) \geq 5$. In order to prove $\chi_{at}(P_2 \times P_n) = 5$, we only prove that $P_2 \times P_n$ has a 5-AVDTC. We construct a mapping $f$ from $V(P_2 \times P_n) \cup E(P_2 \times P_n)$ to $\{1, 2, 3, 4, 5\}$ as follows:

$f(w_{ij}) \in \{1, 2, 3, 4\}$, and $f(w_{ij}) \equiv i + j - 1 \text{ (mod } 4)$, $i = 1, 2, j = 1, 2, \ldots, n$;
\[ f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\}, \text{ and } f(w_{ij}w_{i,j+1}) \equiv i + j + 1 \pmod{4}, i = 1, 2, j = 1, 2, \ldots, n - 1; \]
\[ f(w_{12}w_{2}) = 5, j = 1, 2, \ldots, n. \]
Obviously, \( f \) is a 5-proper-total-coloring. For \( j = 2, 3, \ldots, n - 1 \), we have
\[ C_f(w_{1j}) = \{1\}, j \equiv 2 \pmod{4}; C_f(w_{1j}) = \{2\}, j \equiv 3 \pmod{4}; C_f(w_{1j}) = \{3\}, j \equiv 0 \pmod{4}; \]
\[ C_f(w_{2j}) = \{4\}, j \equiv 1 \pmod{4}. \]
And \( C_f(w_{11}) \neq C_f(w_{21}), C_f(w_{1n}) \neq C_f(w_{2n}) \). So \( f \) is a 5-\textit{AVDTDC}.

**Case 3** \( m = n = 3 \).

In this case, there exists only one vertex of maximum degree (\( =4 \)). So \( \chi_{at}(P_3 \times P_3) \geq 5 \). To prove \( \chi_{at}(P_3 \times P_3) = 5 \), we only prove that \( P_3 \times P_3 \) has a 5-\textit{AVDTDC}. we construct a mapping \( f \) from \( V(P_3 \times P_3) \cup E(P_3 \times P_3) \) to \( \{1, 2, 3, 4, 5\} \) as follows:
\[ f(w_{ij}) \in \{1, 2, 3\}, \text{ and } f(w_{ij}) \equiv i + j - 1 \pmod{3}, i = 1, 2, j = 1, 2, 3; \]
\[ f(w_{ij}w_{i,j+1}) \in \{1, 2, 3\}, \text{ and } f(w_{ij}w_{i,j+1}) \equiv i + j + 1 \pmod{3}, i = 1, 2, 3; j = 1, 2; \]
\[ f(w_{12}w_{2}) = 4, f(w_{23}w_{3}) = 5, j = 1, 2, 3. \]
Obviously, \( f \) is a 5-proper-total-coloring. For every \( xy \in E(P_3 \times P_3) \), we have \( d(x) \neq d(y) \).
So \( f \) is a 5-\textit{AVDTDC}.

**Case 4** \( m = 3, n \geq 4 \text{ or } 4 \leq m \leq n \).

In this case, there exist two adjacent vertices of maximum degree (\( =4 \)). So \( \chi_{at}(P_m \times P_n) \geq 6 \). To prove \( \chi_{at}(P_m \times P_n) = 6 \), we only prove that \( P_m \times P_n \) has a 6-\textit{AVDTDC}. we construct a mapping \( f \) from \( V(P_m \times P_n) \cup E(P_m \times P_n) \) to \( \{1, 2, 3, 4, 5, 6\} \) as follows:
\[ f(w_{ij}) \in \{1, 2, 3, 4\}, \text{ and } f(w_{ij}) \equiv i + j - 1 \pmod{4}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n; \]
\[ f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\}, \text{ and } f(w_{ij}w_{i,j+1}) \equiv i + j + 1 \pmod{4}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n - 1; \]
\[ f(w_{ij}w_{i+1,j}) = 5, j = 1, 2, \ldots, n, i = 1, 2, \ldots, m - 1, i \text{ is odd}; \]
\[ f(w_{ij}w_{i+1,j}) = 6, j = 1, 2, \ldots, n, i = 1, 2, \ldots, m - 1, i \text{ is even}. \]
Obviously, \( f \) is a 6-proper-total-coloring of \( P_m \times P_n \).

And for \( j = 2, 3, \ldots, n - 1 \), we have
\[ C_f(w_{1j}) = \{2, 3, 4, 5\}, j \equiv 2 \pmod{4}; C_f(w_{1j}) = \{3, 4, 1, 5\}, j \equiv 3 \pmod{4}; \]
\[ C_f(w_{2j}) = \{4, 1, 2, 5\}, j \equiv 0 \pmod{4}; C_f(w_{2j}) = \{1, 2, 3, 5\}, j \equiv 1 \pmod{4}. \]
For \( j = 2, 3, \ldots, n - 1 \), we have
\[ C_f(w_{mj}) = \{a(m + j - 1), a(m + j), a(m + j + 1), 5\}, m = \text{even}; \]
\[ C_f(w_{mj}) = \{a(m + j - 1), a(m + j), a(m + j + 1), 6\}, m = \text{even}, \]
where \( a(z) \in \{1, 2, 3, 4\}, a(z) \equiv z \pmod{4} \) for integer \( z \). For \( i = 2, 3, \ldots, m - 1 \), we have
\[ C_f(w_{i1}) = \{2, 4, 5, 6\}, i \text{ is even}; C_f(w_{i1}) = \{3, 1, 5, 6\}, i \text{ is odd}; C_f(w_{im}) = \{a(i + n - 1), a(i + n), 5, 6\}. \]
For \( i = 2, 3, \ldots, m - 1, j = 2, 3, \ldots, n - 1 \), we have that \( C_f(w_{ij}) = \{a(i + j - 1), a(i + j), a(i + j + 1), 5, 6\} \). By careful examination, we can get that for every two adjacent vertices \( x \) and \( y \) of \( P_m \times P_n \), \( C(x) \neq C(y) \). So \( f \) is a 6-\textit{AVDTDC}. The proof is completed. \( \square \)
3. The adjacent-vertex-distinguishing total chromatic number for $P_m \times C_n$

**Theorem 2** Let $m \geq 2, n \geq 3$. Then $\chi_{at}(P_m \times C_n) = \begin{cases} 5, & m = 2; \\ 6, & m \geq 3. \end{cases}$

**Proof** Assume that $P_m = u_1u_2 \cdots u_m, C_n = v_1v_2 \cdots v_nv_1$, and $V(P_m \times C_n) = \{w_{ij} | i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$, $E(P_m \times C_n) = \{w_{ij}w_{rs} | w_{ij}, w_{rs} \in V(P_m \times C_n), i = r, u_ju_s \in E(P_n) \text{ or } j = s, v_iw_r \in E(C_m)\}$. If $r > m, s > n$, then we assume that $w_{rs} = w_{ij} = w_{ij} = v_{ik},$ where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n,$ and $i \equiv r \pmod{m}, j \equiv s \pmod{n}$. There are three cases to be considered.

**Case 1** $m = 2, n = 3$.

In this case, there are two adjacent vertices of maximum degree (=3). So $\chi_{at}(P_2 \times C_3) \geq 5$. To prove $\chi_{at}(P_2 \times C_3) = 5$, we only prove that $P_2 \times C_3$ has a 5-AVDT. We construct a mapping $f$ from $V(P_2 \times C_3) \cup E(P_2 \times C_3)$ to $\{1, 2, 3, 4, 5\}$ as follows:

- $f(w_{11}) = 1, f(w_{12}) = 2, f(w_{13}) = 3, f(w_{22}) = 1, f(w_{23}) = 5; f(w_{11}w_{12}) = 3, f(w_{12}w_{13}) = 4, f(w_{13}w_{11}) = 2; f(w_{21}w_{22}) = 4, f(w_{22}w_{23}) = 2, f(w_{23}w_{21}) = 3; f(w_{11}w_{12}) = 5, f(w_{12}w_{22}) = 5, f(w_{13}w_{23}) = 1.$

We may easily verify that $f$ is a 5-proper-total-coloring. And $C_f(w_{11}) = C_f(w_{21}) = \{1, 2, 3, 5\}; C_f(w_{12}) = C_f(w_{22}) = \{2, 3, 4, 5\}; C_f(w_{13}) = \{1, 2, 3, 4\};$ Thus for arbitrary $xy \in E(P_2 \times C_3)$, we have $C(x) \neq C(y)$. So $f$ is a 5-AVDT.

**Case 2** $m \geq 3, n = 3$.

In this case, there are two adjacent vertices of maximum degree (=4). So $\chi_{at}(P_m \times C_3) \geq 6$ according to Lemma 1. To prove $\chi_{at}(P_m \times C_3) = 6$, we only prove that $P_m \times C_3$ has a 6-AVDT. We construct a mapping $f$ from $V(P_m \times C_3) \cup E(P_m \times C_3)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

- $f(w_{i1}) \in \{1, 2, 3\}$ and $f(w_{i1}) \equiv i \pmod{3}, i = 1, 2, \ldots, m; f(w_{i1}w_{i+1,1}) \in \{1, 2, 3\}$ and $f(w_{i1}w_{i+1,1}) \equiv i + 2 \pmod{3}, i = 1, 2, \ldots, m - 1. f(w_{ij}) \in \{1, 2, 3\}$ and $f(w_{ij}) \equiv f(w_{ij}) + j - 1 \pmod{3}, j = 2, 3, i = 1, 2, \ldots, m; f(w_{ij}w_{i+1,j}) \in \{1, 2, 3\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{ij}w_{i+1,j}) + j - 1 \pmod{3}, j = 2, 3, i = 1, 2, \ldots, m - 1; f(w_{11}w_{12}) = 4, f(w_{12}w_{13}) = 5, f(w_{13}w_{11}) = 6; f(w_{ij}w_{i,j+1}) \in \{4, 5, 6\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{ij}w_{i,j+1}) + i - 1 \pmod{3}, i = 2, 3, \ldots, m; j = 1, 2, 3; So f is a 6-proper-total-coloring.$

Note that $C(w_{11})$ does not contain 5, but contains 4 and 6; $C(w_{12})$ does not contain 6, but contains 4 and 5; $C(w_{13})$ does not contain 4, but contains 5 and 6. One of $C(w_{m1}), C(w_{m2})$ and $C(w_{m3})$ does not contain 4, but contains 5 and 6; Another does not contain 5, but contains 4 and 6; And the third one does not contain 6, but contains 5 and 4. If $C(w_{ij})$ does not contain 4 (5 or 6), then $C(x)$ must contain 4 (5 or 6) for $i = 3, 4, \ldots, m - 2, j = 1, 2, 3,$ and $w_{ij}x \in E(P_m \times C_3)$. So $f$ is a 6-AVDT.

**Case 3** $m \geq 2, n \geq 4$.

In this case, $\chi_{at}(P_m \times C_n) \geq 5$ if $m = 2$ and $\chi_{at}(P_m \times C_n) \geq 6$ if $m \geq 3$ according to Lemma
1. To prove $\chi_{at}(P_m \times C_n) = 5$ when $m = 2$ or $\chi_{at}(P_m \times C_n) = 6$ when $m \geq 3$, we only prove that $P_m \times C_n$ has a 5-AV DTC when $m = 2$ or 6-AV DTC when $m \geq 3$. We construct a mapping $f$ from $V(P_m \times C_n) \cup E(P_m \times C_n)$ to $\{1, 2, 3, 4, 5\}$ when $m = 2$ or $\{1, 2, 3, 4, 5, 6\}$ when $m \geq 3$ as follows.

Firstly, we give a 4-AV DTC for $(P_m \times C_n)[w_1, w_2, \ldots, w_n]$, which is an $n$-cycle induced by the vertices $w_1, w_2, \ldots, w_n$.

If $n \equiv 0 \pmod{4}$, then let
\[ f(w_{1, i+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, i+1}) \equiv i \pmod{4}, i = 1, 2, \ldots, n; \]
\[ f(w_{1, j}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n. \]

If $n \equiv 1 \pmod{4}$, then let
\[ f(w_{1, i+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, i+1}) \equiv i \pmod{4}, i = 1, 2, \ldots, n-5; \]
\[ f(w_{1, j}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n-5; \]
\[ f(w_{1, n-4}w_{1, n-3}) = 1, f(w_{1, n-3}w_{1, n-2}) = 2, f(w_{1, n-2}w_{1, n-1}) = 3, f(w_{1, n}w_{1, n+1}) = 4; \]
\[ f(w_{1, n-1}) = 2, f(w_{1, n-3}) = 3, f(w_{1, n-2}) = 1, f(w_{1, n-1}) = 2, f(w_{1, n}) = 1. \]

If $n \equiv 2 \pmod{4}$, then we let
\[ f(w_{1, i+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, i+1}) \equiv i \pmod{4}, i = 1, 2, \ldots, n-6; \]
\[ f(w_{1, j}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n-6; \]
\[ f(w_{1, n-5}w_{1, n-4}) = 1, f(w_{1, n-4}w_{1, n-3}) = 2, f(w_{1, n-3}w_{1, n-2}) = 3, \]
\[ f(w_{1, n-2}w_{1, n-1}) = 4, f(w_{1, n-1}w_{1, n}) = 3, f(w_{1, n}w_{1, n+1}) = 4; \]
\[ f(w_{1, n-5}) = 2, f(w_{1, n-4}) = 3, f(w_{1, n-3}) = 4, f(w_{1, n-2}) = 1, f(w_{1, n-1}) = 2, f(w_{1, n}) = 1. \]

If $n \equiv 3 \pmod{4}$, then we let
\[ f(w_{1, i+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, i+1}) \equiv i \pmod{4}, i = 1, 2, \ldots, n-7; \]
\[ f(w_{1, j}) \in \{1, 2, 3, 4\} \text{ and } f(w_{1, j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n-7; \]
\[ f(w_{1, n-6}w_{1, n-5}) = 1, f(w_{1, n-5}w_{1, n-4}) = 2, f(w_{1, n-4}w_{1, n-3}) = 3, f(w_{1, n-3}w_{1, n-2}) = 1, \]
\[ f(w_{1, n-2}w_{1, n-1}) = 4, f(w_{1, n-1}w_{1, n}) = 3, f(w_{1, n}w_{1, n+1}) = 4; \]
\[ f(w_{1, n-6}) = 2, f(w_{1, n-5}) = 3, f(w_{1, n-4}) = 4, f(w_{1, n-3}) = 2, \]
\[ f(w_{1, n-2}) = 3, f(w_{1, n-1}) = 2, f(w_{1, n}) = 1. \]

In all above 4 situations, $f$ is a 4-AV DTC of $(P_m \times C_n)[w_1, w_2, \ldots, w_n]$. Secondly, we extend $f$. For $i = 2, 3, \ldots, m$, let
\[ f(w_{ij}) \in \{1, 2, 3, 4\} \text{ and } f(w_{ij}) \equiv f(w_{ij}) \pmod{4}, j = 1, 2, \ldots, n, i \text{ is odd}; \]
\[ f(w_{ij}) \in \{1, 2, 3, 4\} \text{ and } f(w_{ij}) \equiv f(w_{ij}) \pmod{4}, j = 1, 2, \ldots, n, i \text{ is even}; \]
\[ f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{ij}w_{i,j+1}) \equiv f(w_{ij}w_{i,j+1}) \pmod{4}, j = 1, 2, \ldots, n, i \text{ is odd}; \]
\[ f(w_{ij}w_{i,j+1}) \in \{1, 2, 3, 4\} \text{ and } f(w_{ij}w_{i,j+1}) \equiv f(w_{ij}w_{i,j+2}) \pmod{4}, j = 1, 2, \ldots, n, i \text{ is even}. \]

For all $i = 1, 2, \ldots, m - 1$, $j = 1, 2, \ldots, n$, we let
\[ f(w_{ij}w_{i+1,j}) = 5 \text{ when } i \text{ is odd}; f(w_{ij}w_{i+1,j}) = 6 \text{ when } i \text{ is even}. \]

By simple verification, we know that $f$ is a 5-AV DTC when $m = 2$ or 6-AV DTC when $m \geq 3$. The proof is completed. \qed
4. The AVDTC number for $C_m \times C_n$

**Theorem 3** Let $m \geq 3, n \geq 3$. Then $\chi_{at}(C_m \times C_n) = 6$.

**Proof** Assume that $C_m = u_1 u_2 \cdots u_m u_1, C_n = v_1 v_2 \cdots v_n v_1$, and

$$V(C_m \times C_n) = \{w_{ij}| i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\},$$

$$E(C_m \times C_n) = \{w_{ij}w_{rs}| w_{ij}, w_{rs} \in V(C_m \times C_n), i = r, v_j \in E(C_n) \text{ or } j = s, u_i \in E(C_m)\}.$$ 

If $r > m, s > n$, then we assume that $w_{rs} = w_{ij} = w_{rj} = w_{is}$, where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and $i \equiv r \pmod{m}, j \equiv s \pmod{n}$.

Obviously, $\chi_{at}(C_m \times C_n) \geq 6$. To prove $\chi_{at}(C_m \times C_n) = 6$, we only prove that $C_m \times C_n$ has a 6-AVDTC. There are three cases to be considered.

**Case 1** One of $m, n$ is 3.

Without loss of generality, we assume $m = 3$. There are three subcases to be considered in the following.

**Case 1.1** $n \equiv 0 \pmod{3}$.

we construct a mapping $f$ from $V(C_3 \times C_n) \cup E(C_3 \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$f(w_{1j}) \in \{1, 2, 3\}$ and $f(w_{1j}) \equiv j \pmod{3}, j = 1, 2, \ldots, n$;

$f(w_{1j}w_{1,j+1}) \in \{1, 2, 3\}$ and $f(w_{1j}w_{1,j+1}) \equiv j + 2 \pmod{3}, j = 1, 2, \ldots, n$.

$f(w_{ij}) \in \{1, 2, 3\}$ and $f(w_{ij}) \equiv f(w_{1j}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n$;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n$.

$f(w_{11}w_{21}) = 4, f(w_{21}w_{31}) = 5, f(w_{31}w_{11}) = 6$;

$f(w_{ij}w_{i+1,j}) \in \{4, 5, 6\}$ and $f(w_{ij}w_{i+1,j}) \equiv f(w_{11}w_{i+1,1}) + j - 1 \pmod{3}, i = 1, 2, 3; j = 2, 3, \ldots, n$.

Obviously, $f$ is a 6-proper-total-coloring. And

$\overline{\chi}(w_{1j}) = \{5\}, j \equiv 1 \pmod{3}; f(w_{1j}) = \{6\}, j \equiv 2 \pmod{3}; f(w_{1j}) = \{4\}, j \equiv 0 \pmod{3}$;

$f(w_{2j}) = \{6\}, j \equiv 1 \pmod{3}; f(w_{2j}) = \{4\}, j \equiv 2 \pmod{3}; f(w_{2j}) = \{5\}, j \equiv 0 \pmod{3}$;

$f(w_{3j}) = \{4\}, j \equiv 1 \pmod{3}; f(w_{3j}) = \{5\}, j \equiv 2 \pmod{3}; f(w_{3j}) = \{6\}, j \equiv 0 \pmod{3}$. So $f$ is a 6-AVDTC.

**Case 1.2** $n \equiv 1 \pmod{3}$.

we construct a mapping $f$ from $V(C_3 \times C_n) \cup E(C_3 \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows:

$f(w_{1j}) \in \{1, 2, 3\}$ and $f(w_{1j}) \equiv j \pmod{3}, j = 1, 2, \ldots, n - 3$;

$f(w_{1j}w_{1,j+1}) \in \{1, 2, 3\}$ and $f(w_{1j}w_{1,j+1}) \equiv j + 2 \pmod{3}, j = 1, 2, \ldots, n - 3$.

$f(w_{ij}) \in \{1, 2, 3\}$ and $f(w_{ij}) \equiv f(w_{1j}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n - 3$;

$f(w_{ij}w_{i,j+1}) \in \{1, 2, 3\}$ and $f(w_{ij}w_{i,j+1}) \equiv f(w_{1j}w_{1,j+1}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n - 3$.

$f(w_{11}w_{21}) = 4, f(w_{21}w_{31}) = 5, f(w_{31}w_{11}) = 6$;
\[ f(w_{ij} w_{i+1,j}) \in \{4, 5, 6\} \text{ and } f(w_{ij} w_{i+1,j}) \equiv f(w_{i1} w_{i+1,1}) + j - 1 \pmod{3}, \ i = 1, 2, 3; j = 2, 3, \ldots, n - 3; \]
\[ f(w_{1,n-2}) = 2, f(w_{1,n-1}) = 4, f(w_{1n}) = 3, f(w_{2,n-2}) = 3, f(w_{2,n-1}) = 5; \]
\[ f(w_{2n}) = 1, f(w_{3,n-2}) = 4, f(w_{3,n-1}) = 6, f(w_{3n}) = 2; \]
\[ f(w_{1,n-2}w_{1n,1}) = 5, f(w_{1n,1}) = 1, f(w_{1n,11}) = 2; \]
\[ f(w_{2,n-2}w_{2n,1}) = 6, f(w_{2n,1}) = 2, f(w_{2n,21}) = 3; \]
\[ f(w_{3,n-2}w_{3n,1}) = 1, f(w_{3n,1}) = 3, f(w_{3n,31}) = 1; \]
\[ f(w_{1,n-2}w_{2n,2}) = 4, f(w_{2,n-2}w_{3n,2}) = 5, f(w_{3,n-2}w_{3n,2}) = 6; \]
\[ f(w_{1,n-1}w_{2n,2}) = 3, f(w_{2,n-1}w_{3n,2}) = 4, f(w_{3,n-1}w_{3n,2}) = 2; \]
\[ f(w_{1n,2}) = 6, f(w_{2n,3}) = 4, f(w_{3n,1}) = 5. \]

We may verify that \( f \) is a 6-proper-total-coloring. And
\[ \overrightarrow{C}_f(w_{1j}) = \{5\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{1j}) = \{6\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{1j}) = \{4\}, j \equiv 0 \pmod{3}; \]
\[ \overrightarrow{C}_f(w_{2j}) = \{6\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{2j}) = \{4\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{2j}) = \{5\}, j \equiv 0 \pmod{3}; \]
\[ \overrightarrow{C}_f(w_{3j}) = \{4\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{3j}) = \{5\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{3j}) = \{6\}, j \equiv 0 \pmod{3}. \]

\[ \overrightarrow{C}_f(w_{1n,2}) = \{1\}, \overrightarrow{C}_f(w_{1n,1}) = \{6\}, \overrightarrow{C}_f(w_{1n}) = \{4\}, \overrightarrow{C}_f(w_{2n,2}) = \{2\}, \overrightarrow{C}_f(w_{2n,1}) = \{1\}, \overrightarrow{C}_f(w_{2n}) = \{5\}, \overrightarrow{C}_f(w_{3n,2}) = \{3\}, \overrightarrow{C}_f(w_{3n,1}) = \{5\}, \overrightarrow{C}_f(w_{3n}) = \{6\}. \]

So \( f \) is a 6-AV DTC.

**Case 1.3** \( n \equiv 1 \pmod{3} \)

we construct a mapping \( f \) from \( V(C_3 \times C_n) \cup E(C_3 \times C_n) \) to \( \{1, 2, 3, 4, 5, 6\} \) as follows:
\[ f(w_{1i}) \in \{1, 2, 3\} \text{ and } f(w_{1i}) \equiv j \pmod{3}, j = 1, 2, \ldots, n - 4; \]
\[ f(w_{ij} w_{1,j+1}) \in \{1, 2, 3\} \text{ and } f(w_{ij} w_{1,j+1}) \equiv j + 2 \pmod{3}, j = 1, 2, \ldots, n - 4. \]
\[ f(w_{ij}) \in \{1, 2, 3\} \text{ and } f(w_{ij}) \equiv f(w_{ij}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n - 4; \]
\[ f(w_{ij} w_{i,j+1}) \in \{1, 2, 3\} \text{ and } f(w_{ij} w_{i,j+1}) \equiv f(w_{ij} w_{i,j+1}) + j - 1 \pmod{3}, i = 2, 3, j = 1, 2, \ldots, n - 4; \]
\[ f(w_{11} w_{21}) = 4, f(w_{21} w_{31}) = 5, f(w_{31} w_{11}) = 6; \]
\[ f(w_{i1} w_{i+1,j}) \in \{4, 5, 6\} \text{ and } f(w_{i1} w_{i+1,j}) \equiv f(w_{i1} w_{i+1,1}) + j - 1 \pmod{3}, i = 1, 2, 3; j = 2, 3, \ldots, n - 4; \]
\[ f(w_{11} w_{11}) = 2, f(w_{1,n-2}) = 6, f(w_{1,n-1}) = 4, f(w_{1n}) = 3, f(w_{2,n-2}) = 3, f(w_{2,n-1}) = 4, \]
\[ f(w_{2n}) = 1, f(w_{3,n-3}) = 4, f(w_{3,n-2}) = 5, f(w_{3n}) = 6, f(w_{3n,31}) = 2, \]
\[ f(w_{1,n-3}w_{1n,1}) = 5, f(w_{1n,1}) = 1, f(w_{1n,11}) = 2; \]
\[ f(w_{2,n-3}w_{2n,1}) = 6, f(w_{2n,1}) = 2, f(w_{2n,21}) = 3; \]
\[ f(w_{3,n-3}w_{3n,1}) = 1, f(w_{3n,1}) = 3, f(w_{3n,31}) = 1; \]
\[ f(w_{1,n-3}w_{3n,2}) = 4, f(w_{2,n-3}w_{3n,3}) = 5, f(w_{3,n-3}w_{3n,3}) = 6, f(w_{1,n-2}w_{2n,2}) = 1; \]
\[ f(w_{2,n-2}w_{3n,2}) = 2, f(w_{3,n-2}w_{3n,2}) = 3, f(w_{1,n-1}w_{2n,2}) = 6, f(w_{2,n-1}w_{3n,2}) = 1; \]
\[ f(w_{3,n-1}w_{3n,1}) = 5, f(w_{3n,1}) = 6, f(w_{3n,3}) = 4, f(w_{3n,1}) = 5. \]

We may verify that \( f \) is a 6-proper-total-coloring. And
\[ \overrightarrow{C}_f(w_{1j}) = \{5\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{1j}) = \{6\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{1j}) = \{4\}, j \equiv 0 \pmod{3}; \]
\[ \overrightarrow{C}_f(w_{2j}) = \{6\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{2j}) = \{4\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{2j}) = \{5\}, j \equiv 0 \pmod{3}; \]
\[ \overrightarrow{C}_f(w_{3j}) = \{4\}, j \equiv 1 \pmod{3}; \overrightarrow{C}_f(w_{3j}) = \{5\}, j \equiv 2 \pmod{3}; \overrightarrow{C}_f(w_{3j}) = \{6\}, j \equiv 0 \pmod{3}. \]
Case 3 One of $m, n$ is even.

Without loss of generality, we assume that $m$ is even. We construct a mapping $f$ from $V(C_m \times C_n) \cup E(C_m \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows.

Firstly, similar to Case 3 of the proof of Theorem 3, we can give a 4-AVDTD $f$ for $(C_m \times C_n)[w_{11}, w_{12}, \ldots, w_{1n}]$, which is a cycle induced by the vertices $w_{11}, w_{12}, \ldots, w_{1n}$.

Secondly, we extend $f$. Let

- $f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) = f(w_{ij}), i = 2, 3, \ldots, m, j = 1, 2, \ldots, n, i$ is odd;
- $f(w_{ij}) \in \{1, 2, 3, 4\}$ and $f(w_{ij}) = f(w_{1j+1}), i = 2, 3, \ldots, m, j = 1, 2, \ldots, n, i$ is even.

If $f(w_{ij}w_{i+1,j}) = 5$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, i$ is odd; $f(w_{ij}w_{i+1,j}) = 6, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, i$ is even.

We may easily verify that $f$ is a 6-AVDTD of $C_m \times C_n$.

Case 3.1 $n \equiv 1 \pmod{4}$.

We construct a mapping $f$ from $V(C_m \times C_n) \cup E(C_m \times C_n)$ to $\{1, 2, 3, 4, 5, 6\}$ as follows. Let

- $f(w_{1j}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 4$;
- $f(w_{1j}w_{1,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{1j}w_{1,j+1}) \equiv j \pmod{4}, j = 1, 2, \ldots, n - 4$.

If $f(w_{1n-3}w_{1,n-2}) = 2, f(w_{1n-2}w_{1,n-1}) = 4, f(w_{1n-1}w_{1,n}) = 3, f(w_{1n}w_{11}) = 4$.

If $f(w_{2j}) \in \{1, 2, 3, 4\}$ and $f(w_{2j}) \equiv j + 2 \pmod{4}, j = 1, 2, \ldots, n - 4$;

If $f(w_{2j}w_{2,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{2j}w_{2,j+1}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 4$.

If $f(w_{2,j-3}w_{2,j-2}) = 4, f(w_{2,j-2}w_{2,j-1}) = 3, f(w_{2j-1}w_{2j}) = 2$.

If $f(w_{m1}) \in \{1, 2, 3, 4\}$ and $f(w_{m1}) \equiv j + 3 \pmod{4}, j = 1, 2, \ldots, n - 4$;

If $f(w_{m1}w_{m,j+1}) \in \{1, 2, 3, 4\}$ and $f(w_{m1}w_{m,j+1}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 4$.

If $f(w_{m,n-3}w_{m,n-2}) = 6, f(w_{m,n-2}w_{m,n-1}) = 3, f(w_{m,n-1}w_{m,n}) = 5, f(w_{m,n}w_{m1}) = 1$.

If $i = 3, 4, \ldots, m - 1, j = 1, 2, \ldots, n$, let

- $f(w_{ij}) = f(w_{1j})$, if $i$ is odd; $f(w_{ij}) = f(w_{2j})$, if $i$ is even;
- $f(w_{ij}w_{i+1,j+1}) = f(w_{ij}w_{1j+1}), i$ is odd; $f(w_{ij}w_{i+1,j+1}) = f(w_{2j}w_{2,j+1}), i$ is even.

For $i = 1, 2, \ldots, m - 2, j = 1, 2, \ldots, n - 2$, let $f(w_{ij}w_{i+1,j}) \in \{5, 6\}, f(w_{ij}w_{i+1,j}) \equiv i + j - 1 \pmod{2}$. For $i = 1, 2, \ldots, m - 2, j = n - 1, n$, let $f(w_{ij}w_{i+1,j}) \in \{5, 6\}, f(w_{ij}w_{i+1,j}) \equiv i + j \pmod{2}$.

And let

- $f(w_{m-1,j}w_{m,j}) \in \{5, 6\}, f(w_{m-1,j}w_{m,j}) \equiv j + 1 \pmod{2}, j = 1, 2, \ldots, n - 3$;
- $f(w_{m-1,n-2}w_{m,n-2}) = 4, f(w_{m-1,n-1}w_{m,n-1}) = 6, f(w_{m-1,n}w_{m,n}) = 5$. 


\( f(m, jw_1) \in \{1, 2, 3, 4\}, \quad f(w, m, jw_1) \equiv j + 2 \pmod{4}, j = 1, 2, \ldots, n - 3; \)
\( f(w, m, n - 2w_1, n - 2) = 6, \quad f(w, m, n - 1w_1, n - 1) = 1, \quad f(w, m, nw_1n) = 2. \)

We may verify that \( f \) is a 6-proper-total-coloring. Let \( B_i = (\overline{C}_f(w_1), \overline{C}_f(w_2), \ldots, \overline{C}_f(w_m)), i = 1, 2, \ldots, m. \) We have
\[
B_1 = (6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 3, 6, 5); \\
B_2 = (4, 1, 2, 3, 4, 1, 2, 3, \ldots, 4, 1, 2, 3, 4, 1, 2, 3); i = 2, 3, \ldots, m - 2, i \text{ is even}; \\
B_3 = (3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 3, 1, 2), i = 2, 3, \ldots, m - 2, i \text{ is odd}; \\
B_{m-1} = (4, 1, 2, 3, 4, 1, 2, 3, \ldots, 4, 1, 2, 3, 4, 1, 6, 2, 3), \\
B_m = (5, 6, 5, 6, 5, 6, 5, \ldots, 5, 6, 5, 6, 5, 3, 5, 3, 6). \\
\]

So \( f \) is a 6-\( AVDT\) of \( C_m \times C_n \).

**Case 3.2** \( n \equiv 3 \pmod{4} \)

We construct a mapping \( f \) from \( V(C_m \times C_n) \cup E(C_m \times C_n) \) to \( \{1, 2, 3, 4, 5, 6\} \) as follows. Let
\( f(w_{1j}) \in \{1, 2, 3, 4\} \) and \( f(w_{1j}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 6; \)
\( f(w_{1j}w_{1j+1}) \in \{1, 2, 3, 4\} \) and \( f(w_{1j}w_{1j+1}) \equiv j \pmod{4}, j = 1, 2, \ldots, n - 6. \)
\( f(w_{1j}w_{1j-5}) = 3, f(w_{1j}w_{1j-4}) = 4, f(w_{1j}w_{1j-3}) = 2, f(w_{1j}w_{1j-2}) = 3, f(w_{1j}w_{1j-1}) = 2, f(w_{1j}w_{1j}) = 1; \)
\( f(w_{1j}w_{1j-5}w_{1j-4}) = 2, f(w_{1j}w_{1j-4}w_{1j-3}) = 3, f(w_{1j}w_{1j-3}w_{1j-2}) = 1, \)
\( f(w_{1j}w_{1j-2}w_{1j-1}) = 4, f(w_{1j}w_{1j-1}w_{1j}) = 3, f(w_{1j}w_{1j+1}) = 4. \)
\( f(w_{2j}) \in \{1, 2, 3, 4\} \) and \( f(w_{2j}) \equiv j + 2 \pmod{4}, j = 1, 2, \ldots, n - 6; \)
\( f(w_{2j}w_{2j+1}) \in \{1, 2, 3, 4\} \) and \( f(w_{2j}w_{2j+1}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 6. \)
\( f(w_{2j}w_{2j-5}) = 4, f(w_{2j}w_{2j-4}) = 1, f(w_{2j}w_{2j-3}) = 3, f(w_{2j}w_{2j-2}) = 4, f(w_{2j}w_{2j-1}) = 3, f(w_{2j}w_{2j}) = 2; \)
\( f(w_{2j}w_{2j-5}w_{2j-4}) = 3, f(w_{2j}w_{2j-4}w_{2j-3}) = 4, f(w_{2j}w_{2j-3}w_{2j-2}) = 2, \)
\( f(w_{2j}w_{2j-2}w_{2j-1}) = 1, f(w_{2j}w_{2j-1}w_{2j}) = 4, f(w_{2j}w_{2j+1}) = 1. \)
\( f(w_{mj}) \in \{1, 2, 3, 4\} \) and \( f(w_{mj}) \equiv j + 3 \pmod{4}, j = 1, 2, \ldots, n - 6; \)
\( f(w_{mj}w_{mj+1}) \in \{1, 2, 3, 4\} \) and \( f(w_{mj}w_{mj+1}) \equiv j + 1 \pmod{4}, j = 1, 2, \ldots, n - 6. \)
\( f(w_{mn}w_{mn-5}) = 1, f(w_{mn}w_{mn-4}) = 5, f(w_{mn}w_{mn-3}) = 6, f(w_{mn}w_{mn-2}) = 5, f(w_{mn}w_{mn-1}) = 1, f(w_{mn}w_{mn}) = 3; \)
\( f(w_{mn}w_{mn-5}w_{mn-4}) = 3, f(w_{mn}w_{mn-4}w_{mn-3}) = 2, f(w_{mn}w_{mn-3}w_{mn-2}) = 1, \)
\( f(w_{mn}w_{mn-2}w_{mn-1}) = 3, f(w_{mn}w_{mn-1}w_{mn}) = 4, f(w_{mn}w_{mn+1}) = 1. \)

For \( i = 3, 4, \ldots, m - 1, j = 1, 2, \ldots, n, \) let
\( f(w_{ij}) = f(w_{ij}), \) if \( i \) is odd; \( f(w_{ij}) = f(w_{ij}), \) if \( i \) is even;
\( f(w_{ij}w_{ij+1}) = f(w_{ij}w_{ij+1}), \) if \( i \) is odd; \( f(w_{ij}w_{ij+1}) = f(w_{ij}w_{ij+1}), \) if \( i \) is even.

For \( i = 1, 2, \ldots, m - 2, j = 1, 2, \ldots, n - 2, \) let \( f(w_{ij}w_{ij+1}) \in \{5, 6\}, f(w_{ij}w_{ij+1}) \equiv i + j - 1 \pmod{2}. \) For \( i = 1, 2, \ldots, m - 2, j = n - 1, n, \) let \( f(w_{ij}w_{ij+1}) \in \{5, 6\}, f(w_{ij}w_{ij+1}) \equiv i + j \pmod{2}. \) And let
\( f(w_{mn}w_{mn-5}w_{mn-4}) \in \{5, 6\}, f(w_{mn}w_{mn-5}w_{mn-4}) \equiv j + 1 \pmod{2}, j = 1, 2, \ldots, n - 2; \)
\( f(w_{mn}w_{mn-5}w_{mn-4}) \equiv 2, f(w_{mn}w_{mn-5}w_{mn}) = 5 \)
\( f(w_{mn}w_{mn-4}w_{mn-3}) \equiv j + 2 \pmod{4}, j = 1, 2, \ldots, n - 4; \)
\( f(w_{mn}w_{mn-3}w_{mn-2}) = 4, f(w_{mn}w_{mn-2}w_{mn-1}) = 2, f(w_{mn}w_{mn-1}w_{mn}) = 6, f(w_{mn}w_{mn+1}) = 2. \)

We may verify that \( f \) is a 6-proper-total-coloring.

Let \( B_i = (\overline{C}_f(w_1), \overline{C}_f(w_2), \ldots, \overline{C}_f(w_m)), i = 1, 2, \ldots, m. \) We have
\( B_1 = (6, 5, 6, 5, 6, 5, 6, 5, \ldots, 6, 5, 6, 5, 6, 5, 6, 1, 5); \)

\( B_i = (4, 1, 2, 3, 4, 1, 2, 3, \ldots, 4, 1, 2, 3, 4, 1, 2, 3, 2, 3), i = 2, 3, \ldots, m - 2, \) \( i \) is even;

\( B_i = (3, 4, 1, 2, 3, 4, 1, 2, \ldots, 3, 4, 1, 2, 3, 4, 1, 2, 1, 2), i = 2, 3, \ldots, m - 2, \) \( i \) is odd;

\( B_{m-1} = (4, 1, 2, 3, 4, 1, 2, 3, \ldots, 4, 1, 2, 3, 4, 1, 2, 1, 3, 6, 3), \)

\( B_m = (5, 6, 5, 6, 5, 6, 5, \ldots, 5, 6, 5, 6, 5, 6, 4, 3, 3, 4, 5, 6). \)

So \( f \) is a 6-AV DTC of \( C_m \times C_n \).

The proof is completed. \( \square \)

References


