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Quantum Adjoint Action for Quantum Algebra $\mathcal{U}_q(f(K,H))$

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Abstract The aim of this paper is to study the adjoint action for the quantum algebra $\mathcal{U}_q(f(K,H))$, which is a natural generalization of quantum algebra $\mathcal{U}_q(\mathrm{sl}_2)$ and is regarded as a class of generalized Weyl algebra. The structure theorem of its locally finite subalgebra $\mathcal{F}(\mathcal{U}_q(f(K,H)))$ is given.

Keywords adjoint action; locally finite subalgebra; highest weight vector.

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0. Introduction

Most important quantum algebras are the q-deformations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$. And the simplest and most important example is the Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathrm{sl}_2)$, which appeared first in 1983 in a paper by Kulish and Reshtikhin[1] on the study of integrable XYZ module with highest spin and whose Hopf algebra structure was discovered later by Sklyanin[2]. Various generalized (Weyl) algebras of $\mathcal{U}(\mathrm{sl}_2)$ and $\mathcal{U}_q(\mathrm{sl}_2)$ have been studied by many authors[3−6]. In particular, Wang[7] introduced a quantum algebra $\mathcal{U}_q(f(K,H))$ as a natural generalization of $\mathcal{U}_q(\mathrm{sl}_2)$. Moreover, it can be regarded not only as a generalization of Drinfeld double $\mathcal{D}(\mathrm{sl}_2)$[7], but also as a class of generalized Weyl algebras defined by Bavula[4]. Thus studying the structure of $\mathcal{U}_q(f(K,H))$ is a very interesting and significant work. In [7], a necessary and sufficient condition for $\mathcal{U}_q(f(K,H))$ to be a Hopf algebra was given, moreover, finite dimensional representations and the center of $\mathcal{U}_q(f(K,H))$ were discussed. Our main aim in this paper is to discuss the irreducible $\mathcal{U}_q(f(K,H))$-submodules of $\mathcal{U}_q(f(K,H))$ under the adjoint action and give the structure theorem of its locally finite subalgebra $\mathcal{F}(\mathcal{U}_q(f(K,H)))$.

1. Quantum algebra $\mathcal{U}_q(f(K,H))$

Throughout this paper k denotes the complex field and $q\in k\setminus\{0\}$ is not a root of the unity.

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Definition 1.1[7] *Define $\mathcal{U}_q(f(K,H))$ as the algebra generated by E,F,K,H and K^{-1},H^{-1} with the relations*

$$KH=HK,\quad KK^{-1}=K^{-1}K=1,\quad HH^{-1}=H^{-1}H=1,$$

$$KEK^{-1}=q^2E,\quad KFK^{-1}=q^{-2}F,$$

$$HEH^{-1}=q^{-2}E,\quad HFH^{-1}=q^2F,$$

$$[E,F]=EF-FE=f(K,H),$$

where, $f(K,H)=\sum_{i,j=-N}^{N}a_{ij}K^iH^j\in k[K,H,K^{-1},H^{-1}]$ and $N\in\mathbb{Z}^+$.

Set $(n)_q=1+q^2+\cdots+q^{n-1}=\frac{q^n-1}{q-1}$. For any Laurent polynomial

$$g(K,H)=\sum_{i,j=-N}^{N}a_{ij}K^iH^j\in k[K,H,K^{-1}H^{-1}],$$

we define the following notations. For any $s,m\in\mathbb{N}$, set

$$g^{+(s)}(K,H)=\sum_{i,j=-N}^{N}q^{2s(i-j)}a_{ij}K^iH^j,$$

$$g^{-(s)}(K,H)=\sum_{i,j=-N}^{N}q^{-2s(i-j)}a_{ij}K^iH^j,$$

$$g_{+(m)}(K,H)=\sum_{i,j=-N}^{N}(m)_{q^{2(i-j)}}a_{ij}K^iH^j,$$

$$g_{-(m)}(K,H)=\sum_{i,j=-N}^{N}(m)_{-q^{-2(i-j)}}a_{ij}K^iH^j.$$

Then, we have

$$g(K,H)F^s=F^sg^{-(s)}(K,H),\quad F^sg(K,H)=g^{+(s)}(K,H)F^s,$$

$$g_{+(m)}(K,H)=\sum_{s=0}^{m-1}g^{+(s)}(K,H),\quad g_{-(m)}(K,H)=\sum_{s=0}^{m-1}g^{-(s)}(K,H).$$

Moreover, for any $m\in\mathbb{N}$, the following relations hold in $\mathcal{U}_q(f(K,H))$:

$$EF^m-F^mE=F^{m-1}f_{-(m)}(K)=f_{+(m)}(K)F^{m-1},$$

$$E^mF-FE^m=E^{m-1}f_{+(m)}(K)=f_{-(m)}(K)E^{m-1}.$$

The algebra $\mathcal{U}_q(f(K,H))$ is Noetherian and has no zero divisors, and the set $\{E^iF^jK^lH^r\}$ $(i,j\in\mathbb{N},l,r\in\mathbb{Z})$ is its basis.

In what follows, we always assume $f(K)=a(K^mH^n-K^{-m'}H^{-n'})$ for some $a\in k\setminus\{0\}$, and some $m,m',n,n'\in\mathbb{Z}^+$ with $M=m-n=m'-n'$. In this situation, the algebra $\mathcal{U}_q(f(K,H))$ has a Hopf algebra structure: In fact, for some $h,k,s,t,h',k',s',t'\in\mathbb{Z}$,

$$\Delta(K)=K\otimes K,\quad \Delta(K^{-1})=K^{-1}\otimes K^{-1},$$

$$\Delta(H) = H \otimes H, \quad \Delta(H^{-1}) = H^{-1} \otimes H^{-1},$$

$$\Delta(E) = K^s H^t \otimes E + E \otimes K^h H^k,$$

$$\Delta(F) = K^{-h'} H^{-k'} \otimes F + F \otimes K^{-s'} H^{-t'},$$

$$\varepsilon(K) = \varepsilon(K^{-1}) = 1, \quad \varepsilon(H) = \varepsilon(H^{-1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0,$$

$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(H) = H^{-1}, \quad S(H^{-1}) = H,$$

$$S(E) = -K^{-s} H^{-t} E K^{-h} H^{-k}, \quad S(F) = -K^{h'} H^{k'} F K^{s'} H^{t'},$$

where, $M = h + t - k - s$, and $s - t = s' - t', h - k = h' - k'$.

Set

$$f_q^+(K,H) = a\Big(\frac{K^m H^n}{q^{2(m-n)} - 1} - \frac{K^{-m'} H^{-n'}}{q^{-2(m'-n')} - 1}\Big),$$

$$f_q^-(K,H) = a\Big(\frac{K^m H^n}{1 - q^{-2(m-n)}} - \frac{K^{-m'} H^{-n'}}{1 - q^{-2(m'-n')}}\Big).$$

Then the element $C_q(f(K)) = EF + f_q^+(K) = FE + f_q^-(K)$, which is called the Casimir element of $\mathcal{U}_q(f(K,H))$, generates the center of $\mathcal{U}_q(f(K,H))$ as a polynomial algebra.

By [7, Proposition 3.4], for all $i, j \in \mathbb{N}$, we have the following equations:

$$\Delta(F^j) = \sum_{r=0}^{j} \binom{j}{r}_{q^{-2M}} q^{2r(j-r)(h'-k')} F^{j-r} K^{-rh'} H^{-rk'} \otimes F^r K^{-s'(j-r)} H^{-t'(j-r)}, \tag{1.1}$$

$$\Delta(E^i) = \sum_{r=0}^{i} \binom{i}{r}_{q^{2M}} q^{2r(i-r)(s-t)} E^{i-r} K^{rs} H^{rt} \otimes E^r K^{h(i-r)} H^{k(i-r)}. \tag{1.2}$$

Where, $\binom{i}{s}_{q^{2M}}, \binom{j}{r}_{q^{-2M}}$ are the Gauss polynomials (see [8, Chapter 4]).

Definition 1.2 *Let V be a $\mathcal{U}_q(f(K,H))$-module and a, b scalars. An element $v \neq 0$ in V is called a highest (resp. lowest) weight vector of weight $(a,b) \in k \times k$ if $K \cdot v = av$, $H \cdot v = bv$, and if $E \cdot v = 0$ (resp. $F \cdot v = 0$). A $\mathcal{U}_q(f(K,H))$-module is called a highest weight module if it is generated by a highest weight vector.*

Let $d \in \mathbb{N}$. We define a $\mathcal{U}_q(f(K,H))$-module denoted by $V(d)$ as follows: The set $\{v_0, v_1, \ldots, v_d\}$ is its basis and satisfies the following relations:

$K \cdot v_i = q^{d-2i} v_i$ for $0 \le i \le d$, $H \cdot v_i = q^{-d+2i} v_i$ for $0 \le i \le d$,

$F \cdot v_i = v_{i+1}$ for $0 \le i < d$, and $F \cdot v_d = 0$,

$E \cdot v_i = f_{-(i)}(q^d, q^{-d}) v_{i-1}$ for $0 < i \le d$, and $E \cdot v_0 = 0$.

Then $V(d)$ is a highest weight $\mathcal{U}_q(f(K,H))$-module of weight (q^d, q^{-d}) and its dimension is $d+1$. Thus by [7, Theorem 4.5], we know that $V(d)$ is simple, moreover, any $d+1$ dimensional simple $\mathcal{U}_q(f(K,H))$-module is isomorphic to $V(d)$.

2. The locally finite subalgebra of $\mathcal{U}_q(f(K,H))$

Let H be a Hopf algebra over a field k with a comultiplication Δ, a counit ε and an antipode S. We use the Sweedler's notation to denote Δ, i.e., $\Delta(x) = x_1 \otimes x_2$ for all $x \in H$. For a Hopf algebra H and $x, y \in H$, we set $(\mathrm{ad}x)(y) = x_1 y S(x_2)$. Then, the action endows H with the structure of a left module algebra on itself, which is called the left adjoint action of H[8,9]. Let $\mathcal{F}(H)$ denote the set of all elements on which the left adjoint action is locally finite, i.e.,

$$\mathcal{F}(H) = \{x \in H \mid \dim_k(\mathrm{ad}H)x < \infty\},$$

which is a subalgebra and a submodule of H, and is called the locally finite subalgebra of H. As we know, the left adjoint action of H and the locally finite subalgebra $\mathcal{F}(H)$ play important roles in the study of $\mathrm{Prim}(H)$, the set of all prime ideals of H[10,11]. Catoiu[10] studied the structure of $\mathcal{F}(H)$ when H is the universal enveloping algebra $\mathcal{U}(sl_2)$. Li and Zhang[12] studied that of $\mathcal{F}(H)$ when H is the quantized enveloping algebra $\mathcal{U}_q(sl_2)$.

For quantum algebra $\mathcal{U}_q(f(K,H)$, the adjoint actions of generators of $\mathcal{U}_q(f(K,H))$ can be represented as

$$\begin{aligned}
&(\mathrm{ad}K)(x) = KxK^{-1}, \quad (\mathrm{ad}K^{-1})(x) = K^{-1}xK,\\
&(\mathrm{ad}H)(x) = HxH^{-1}, \quad (\mathrm{ad}H^{-1})(x) = H^{-1}xH,\\
&(\mathrm{ad}E)(x) = ExK^{-h}H^{-k} - K^sH^txK^{-s}H^{-t}EK^{-h}H^{-k},\\
&(\mathrm{ad}F)(x) = FxK^{s'}H^{t'} - K^{-h'}H^{-k'}xK^{h'}H^{k'}FK^{s'}H^{t'},
\end{aligned} \tag{2.1}$$

for all $x \in \mathcal{U}_q(f(K,H))$. And the locally finite subalgebra of $\mathcal{U}_q(f(K,H))$, $\mathcal{F}(\mathcal{U}_q(f(K,H))$, is a left $\mathcal{U}_q(f(K,H))$-module algebra and is semisimple. Let $x \in \mathcal{U}_q(f(K,H))$ and set

$$[x] = \mathrm{ad}(\mathcal{U}_q(f(K,H))(x)$$

denoting the $\mathcal{U}_q(f(K,H))$-submodule of $\mathcal{U}_q(f(K,H))$ generated by x.

Proposition 2.1 *For any $d \in \mathbb{N}$, we have $[E^dK^{-ds}H^{-dt}] \cong V(2d)$.*

Proof By [7, Theorem 4.5], we only need prove that $E^dK^{-ds}H^{-dt}$ is a highest weight vector of weight (q^{2d}, q^{-2d}) and the endomorphism induced by F is nilpotent.

In fact, by the equation (2.1), we have

$$(\mathrm{ad}K)(E^dK^{-ds}H^{-dt}) = q^{2d}E^dK^{-ds}H^{-dt}, \quad (\mathrm{ad}H)(E^dK^{-ds}H^{-dt}) = q^{-2d}E^dK^{-ds}H^{-dt},$$

and $(\mathrm{ad}E)(E^dK^{-ds}H^{-dt}) = 0$. Therefore, $E^dK^{-ds}H^{-dt}$ is a highest weight vector of weight (q^{2d}, q^{-2d}).

Now, we prove that the relation $(\mathrm{ad}F^{2d+1})(E^dK^{-ds}H^{-dt}) = 0$ holds for $d \in \mathbb{N}$. First, we consider the situation when $d = 1$. By the equation (2.1), we have

$$\begin{aligned}
(\mathrm{ad}F)(EK^{-s}H^{-t}) &= FEK^{-s}H^{-t}K^{s'}H^{t'} - K^{-h'}H^{-k'}EK^{-s}H^{-t}K^{h'}H^{k'}FK^{s'}H^{t'}\\
&= FEK^{s'-s}H^{t'-t} - q^{2(k'-h'+s-t)}EFK^{s'-s}H^{t'-t}\\
&= FEK^{s'-s}H^{t'-t} - q^{-2M}EFK^{s'-s}H^{t'-t},
\end{aligned}$$

hence

$$\begin{aligned}
&(\mathrm{ad}F^2)(EK^{-s}H^{-t})\\
&\quad = (\mathrm{ad}F)(FEK^{s'-s}H^{t'-t} - q^{-2M}EFK^{s'-s}H^{t'-t})
\end{aligned}$$

$$= F(FEK^{s'-s}H^{t'-t} - q^{-2M}EFK^{s'-s}H^{t'-t})K^{s'}H^{t'} - K^{-h'}H^{-k'}(FEK^{s'-s}H^{t'-t} - q^{-2M}EFK^{s'-s}H^{t'-t})K^{h'}H^{k'}FK^{s'}H^{t'}$$
$$= F(FEK^{s'-s}H^{t'-t} - q^{-2M}EFK^{s'-s}H^{t'-t})K^{s'}H^{t'} - FEK^{s'-s}H^{t'-t}FK^{s'}H^{t'} + q^{-2M}EFK^{s'-s}H^{t'-t}FK^{s'}H^{t'}$$
$$= F(FEK^{2s'-s}H^{2t'-t} - q^{-2M}EFK^{2s'-s}H^{2t'-t}) - q^{2(s-s'+t'-t)}FEFK^{2s'-s}H^{2t'-t} + q^{-2M}q^{t'-t-s'+s}EFFK^{2s'-s}H^{2t'-t}$$
$$= -F(EF-FE)K^{2s'-s}H^{2t'-t} + q^{-2M}(EF-FE)FK^{2s'-s}H^{2t'-t}$$
$$= -Fa(K^mH^n - K^{-m'}H^{-n'})K^{2s'-s}H^{2t'-t} + q^{-2M}a(K^mH^n - K^{-m'}H^{-n'})FK^{2s'-s}H^{2t'-t}$$
$$= -aF(K^mH^n - K^{-m'}H^{-n'})K^{2s'-s}H^{2t'-t} + q^{-2M}aq^{2n-2m}FK^mH^nK^{2s'-s}H^{2t'-t} - q^{-2M}aq^{2m'-2n'}FK^{-m'}H^{-n'}K^{2s'-s}H^{2t'-t}$$
$$= -aF(K^mH^n - K^{-m'}H^{-n'})K^{2s'-s}H^{2t'-t} + q^{-4M}aFK^mH^nK^{2s'-s}H^{2t'-t} - aFK^{-m'}H^{-n'}K^{2s'-s}H^{2t'-t}$$
$$= (q^{-4M}-1)aFK^{2s'-s+m}H^{2t'-t+n},$$

and

$$(\mathrm{ad}F^3)(EK^{-s}H^{-t})$$
$$= (\mathrm{ad}F)(q^{-4M}-1)aFK^{2s'-s+m}H^{2t'-t+n}$$
$$= a(q^{-4M}-1)(FFK^{2s'-s+m}H^{2t'-t+n}K^{s'}H^{t'} - K^{-h'}H^{-k'}FK^{2s'-s+m}H^{2t'-t+n}K^{h'}H^{k'}FK^{s'}H^{t'})$$
$$= a(q^{-4M}-1)(F^2K^{3s'-s+m}H^{3t'-t+n} - q^{2h'-2k'}FK^{2s'-s+m}H^{2t'-t+n}FK^{s'}H^{t'})$$
$$= a(q^{-4M}-1)(F^2K^{3s'-s+m}H^{3t'-t+n} - F^2K^{3s'-s+m}H^{3t'-t+n}) = 0.$$

Then, we assume $d > 1$ and that the relation $(\mathrm{ad}F^{2i+1})(E^iK^{-is}H^{-it}) = 0$ holds for all $i < d$. By the equations (1.1), (2.1) and the assumption, we have

$$(\mathrm{ad}F^{2d+1})(E^dK^{-ds}H^{-dt})$$
$$= q^{2(d-1)(t-s)}(\mathrm{ad}F^{2d+1})(E^{d-1}K^{-(d-1)s}H^{-(d-1)t}EK^{-s}H^{-t})$$
$$= \sum_{r=0}^{2d+1}\binom{2d+1}{r}_{q^{-2M}} q^{2(d-1)(t-s)}q^{2r(2d+1-r)(h'-k')} (\mathrm{ad}(F^{2d+1-r}K^{-rh'}H^{-rk'}))(E^{d-1}K^{-(d-1)s}H^{-(d-1)t}) (\mathrm{ad}(F^rK^{-s'(2d+1-r)}H^{-t'(2d+1-r)}))(EK^{-s}H^{-t})$$
$$= \sum_{r=0}^{2d+1}\binom{2d+1}{r}_{q^{-2M}} q^{2(d-1)(t-s)+2r(2d+1-r)(h'-k')}q^{2r(d-1)(k'-h')}q^{2(2d+1-r)(s'-t')} (\mathrm{ad}(F^{2d+1-r}))(E^{d-1}K^{-(d-1)s}H^{-(d-1)t})(\mathrm{ad}(F^r))(EK^{-s}H^{-t})$$
$$= 0.$$

Hence the action of F on $E^dK^{-ds}H^{-dt}$ is nilpotent and $[E^dK^{-ds}H^{-dt}] \cong V(2d)$. The proof is completed. □

Corollary 2.2 *For any $d \in \mathbb{N}$, $(\mathrm{ad}F^{2d})(E^dK^{-ds}H^{-dt})$ is a lowest weight vector of weight (q^{2d}, q^{-2d}), hence $[(\mathrm{ad}F^{2d})(E^dK^{-ds}H^{-dt})] \cong V(2d)$.*

The following proposition determines all irreducible $\mathcal{U}_q(f(K,H))$-submodules of $\mathcal{F}(\mathcal{U}_q(f(K,H)))$.

Proposition 2.3 *Let V be any irreducible $\mathcal{U}_q(f(K,H))$-submodule of $\mathcal{F}(\mathcal{U}_q(f(K,H)))$. Then $V = [g(C_q)E^dK^{-ds}H^{-dt}]$ for some polynomial $g(C_q) \neq 0$ and $d \geq 0$. Moreover, we have $V \cong [E^dK^{-ds}H^{-dt}]$.*

Proof Suppose V is any irreducible $\mathcal{U}_q(f(K,H))$-submodule of $\mathcal{F}(\mathcal{U}_q(f(K,H)))$ with its dimension $c+1$. Then by [7, Theorems 4.4, 4.5], there exists a highest weight vector v of weight (q^c, q^{-c}) in V. We may assume $v = \sum_{i,j,l,r} E^iF^jK^lH^r$. Considering the equations $(\mathrm{ad}K)v = q^cv$ and $(\mathrm{ad}H)v = q^{-c}v$, we easily get $c = 2d$ and $i = d+j$. So, we can rewrite v as the form

$$\sum_j E^{d+j}F^jg_j(K,H,K^{-1},H^{-1}) = E^d(\sum_j E^jF^jg_j(K,H,K^{-1},H^{-1}))$$

for some Laurent polynomials $g_j(K,H,K^{-1},H^{-1})$.

According to $C_q(f(K)) = EF + f_q^+(K) = FE + f_q^-(K)$, we can rewrite v as the form $E^dh(C_q,K,H,K^{-1},H^{-1})$ for some polynomial $h(C_q,K,H,K^{-1},H^{-1})$. Now we consider the equation $(\mathrm{ad}E)v = 0$. Then we have

$$\begin{aligned}
&EE^dh(C_q,K,H,K^{-1},H^{-1})K^{-h}H^{-k}\\
&\quad = K^sH^tE^dh(C_q,K,H,K^{-1},H^{-1})K^{-s}H^{-t}EK^{-h}H^{-k}\\
&\quad = q^{2d(s-t)}E^dh(C_q,K,H,K^{-1},H^{-1})EK^{-h}H^{-k}.
\end{aligned}$$

Since $\mathcal{U}_q(f(K,H))$ has no zero divisors, we have

$$Eh(C_q,K,H,K^{-1},H^{-1}) = q^{2d(s-t)}h(C_q,K,H,K^{-1},H^{-1})E.$$

Note that C_q belongs to the center of $\mathcal{U}_q(f(K,H))$ and q is not a root of the unity. It follows that $h(C_q,K,H,K^{-1},H^{-1})$ has the form $g(C_q)K^{-ds}H^{-dt}$. Thus $v = g(C_q)E^dK^{-ds}H^{-dt}$, and $[v] = [g(C_q)E^dK^{-ds}H^{-dt}] \cong [E^dK^{-ds}H^{-dt}]$. □

Proposition 2.4 *Suppose $v = \sum_{i=1}^m g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}$, where $g_i(C_q) \neq 0$, and the integers d_i are pairwise distinct. Then*

$$[v] = \bigoplus_{i=1}^m [g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}] \cong \bigoplus_{i=1}^m [E^{d_i}K^{-d_is}H^{-d_it}].$$

Proof Since $g_i(C_q)$ belongs to the center of $\mathcal{U}_q(f(K,H))$ for every i, we have that

$$[g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}] \cong [E^{d_i}K^{-d_is}H^{-d_it}] \cong V(2d_i)$$

by Proposition 2.1. Note that the integers d_i are pairwise distinct. It follows that

$$\sum_{i=1}^{m}[g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}] = \bigoplus_{i=1}^{m}[g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}] \cong \bigoplus_{i=1}^{m}[E^{d_i}K^{-d_is}H^{-d_it}].$$

Clearly, we have $[v] \subseteq \sum_{i=1}^{m}[g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}]$. On the other hand, without loss of generality, we may assume that $d_1 < d_2 < \cdots < d_m$. Then we have

$$(\mathrm{ad}F)^{2d_m}(v) = (\mathrm{ad}F)^{2d_m}(g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt}) \in [v].$$

By Corollary 2.2, $(\mathrm{ad}F)^{2d_m}(g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt})$ is a lowest weight vector with weight (q^{2d_m}, q^{-2d_m}) in the irreducible module $[g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt}]$, which implies that

$$[g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt}] \subseteq [v] \quad g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt} \in [v].$$

Set $u = v - g_m(C_q)E^{d_m}K^{-d_ms}H^{-d_mt} \in [v]$. In the same way, we can prove that

$$g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it} \in [v]$$

for $1 \leq i < m$. Thus, $\sum_{i=1}^{m}[g_i(C_q)E^{d_i}K^{-d_is}H^{-d_it}] \subseteq [v]$. □

By the results obtained above, we can get the following theorem.

Theorem 2.5 *As $\mathcal{U}_q(f(K,H))$-modules, we have*

$$\mathcal{F}(\mathcal{U}_q(f(K,H))) = \bigoplus_{i,j\geq 0}[C_q^jE^iK^{-is}H^{-it}].$$

Corollary 2.6 *Assume that $m = m' = h - s, n = n' = k - t, t = t', s = s'$. Then we have $[K^mH^n] = [EK^{-s}H^{-t}] \oplus [C_q]$.*

Proof On the one hand, from the proof of Proposition 2.1, we know that $[EK^{-s}H^{-t}]$ is spanned by

$$EK^{-s}H^{-t}, \quad FE - q^{-2M}EF, \quad FK^{s+m}H^{t+n}.$$

On the other hand, by assumption we have

$$(\mathrm{ad}K)K^{h-s}H^{k-t} = K^{h-s}H^{k-t},$$
$$(\mathrm{ad}E)K^{h-s}H^{k-t} = EK^{-s}H^{-t} - q^{2(h-s+t-k)}EK^{-s}H^{-t} = (1-q^{2M})EK^{-s}H^{-t},$$
$$(\mathrm{ad}F)K^{h-s}H^{k-t} = FK^hH^k - q^{2(k-t+h-s)}FK^hH^k = (1-q^{-2M})FK^hH^k.$$

Then $[K^{h-s}H^{k-t}]$ is spanned by $K^{h-s}H^{k-t}$, $EK^{-s}H^{-t}$, FK^hH^k, $FE - q^{-2M}EF$. But

$$\begin{aligned}&\frac{-q^M}{a(q^M+q^{-M})}(FE - q^{-2M}EF) + \frac{q^M - q^{-M}}{a(q^M+q^{-M})}C_q\\ &= \frac{-q^M}{a(q^M+q^{-M})}(FE - q^{-2M}EF) +\\ &\quad \frac{q^M - q^{-M}}{a(q^M+q^{-M})}(EF + a(\frac{K^mH^n}{q^{2M}-1} - \frac{K^{-m}H^{-n}}{q^{-2M}-1}))\end{aligned}$$

$$
\begin{aligned}
&= \frac{-q^M}{a(q^M+q^{-M})}(FE-EF-(q^{-2M}-1)EF)+\\
&\quad \frac{q^M-q^{-M}}{a(q^M+q^{-M})}(EF+a(\frac{K^mH^n}{q^{2M}-1}-\frac{K^{-m}H^{-n}}{q^{-2M}-1}))\\
&= \frac{-q^M}{a(q^M+q^{-M})}(FE-EF)+\frac{q^M(q^{-2M}-1)}{a(q^M+q^{-M})}EF+\\
&\quad \frac{q^M-q^{-M}}{a(q^M+q^{-M})}(EF+a(\frac{K^mH^n}{q^{2M}-1}-\frac{K^{-m}H^{-n}}{q^{-2M}-1}))\\
&= \frac{q^M}{a(q^M+q^{-M})}(EF-FE)+\frac{q^M-q^{-M}}{q^M+q^{-M}}(\frac{K^mH^n}{q^{2M}-1}-\frac{K^{-m}H^{-n}}{q^{-2M}-1})\\
&= \frac{q^M}{q^M+q^{-M}}(K^mH^n-K^{-m}H^{-n})+\frac{q^M-q^{-M}}{(q^M+q^{-M})(q^{2M}-1)}K^mH^n-\\
&\quad \frac{q^M-q^{-M}}{(q^M+q^{-M})(q^{-2M}-1)}K^{-m}H^{-n}\\
&= K^mH^n-(\frac{q^M}{q^M+q^{-M}}+\frac{q^M-q^{-M}}{(q^M+q^{-M})(q^{-2M}-1)})K^{-m}H^{-n}\\
&= K^mH^n.
\end{aligned}
$$

Hence, we have

$$
K^mH^n=\frac{-q^M}{a(q^M+q^{-M})}(FE-q^{-2M}EF)+\frac{q^M-q^{-M}}{a(q^M+q^{-M})}C_q,
$$

$$
[K^mH^n]\subseteq[EK^{-s}H^{-t}]\oplus[C_q].
$$

The proof is completed. □

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