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A GLKKM Type Theorem for Noncompact Complete L-Convex Metric Spaces with Applications to Variational Inequalities and Fixed Points

WEN Kai Ting
(Department of Mathematics, Bijie University, Guizhou 551700, China)
(E-mail: wenkaiting_2004@sina.com.cn)

Abstract In this paper, a new GLKKM type theorem is established for noncompact complete L-convex metric spaces. As applications, the properties of the solution set of variational inequalities, intersection point sets, Ky Fan sections and maximal element sets are shown, and a Fan-Browder fixed point theorem is obtained.

Keywords L-convex metric space; noncompact measure; transfer compactly open (closed); variational inequality; section; maximal element; fixed point.

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0. Introduction

In 1956, Aronszajn and Panitchpakdi[1] introduced the notion of hyperconvex metric spaces. Recently, Khamsi[2] established a hyperconvex version of the famous KKM-Fan principle. Yuan[3] studied the characterization for a mapping with finitely metrically open values being a generalized metric KKM mapping in hyperconvex spaces. Park[4] obtained a Ky Fan matching theorem for open covers, a coincidence theorem and other results for hyperconvex spaces. Kirk et al.[5] established KKM theory in hyperconvex spaces and as applications of their results, the fixed point theorem, maximal element theorem, intersection theorem, section theorem and the other results were given in hyperconvex spaces. In [6], we established a Browder fixed point theorem in noncompact admissible subsets of noncompact hyperconvex spaces, which was used to derive two Ky Fan coincidence theorems.

In 2001, Ding and Xia[7] introduced H-metric spaces and generalized H-KKM mappings, and established some generalized H-KKM type theorems for generalized H-KKM mappings in H-metric spaces. In 2005, Meng et al.[8] introduced G-convex metric spaces and established some generalized KKM type theorems and fixed point theorems in G-convex metric spaces. In [9], we first introduced the new notion of L-convex metric spaces, and then established some GLKKM

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type theorems for GLKKM mappings and a Browder fixed point theorem in L-convex metric spaces.

In this paper, a new GLKKM type theorem is established for noncompact complete L-convex metric spaces. As applications, the properties of the solution set of variational inequalities, intersection point sets, Ky Fan sections and maximal element sets are shown, and a Fan-Browder fixed point theorem is obtained. Our results unify, improve and generalize some recent known results in several aspects.

1. Preliminaries

Let X be a nonempty set. We denote by $\mathcal{F}(X)$ and 2^X the family of all nonempty finite subsets of X and the family of all subsets of X, respectively, by $|A|$ the cardinality of A for each $A \in \mathcal{F}(X)$, and by $\triangle_n$ the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. Let D be a subset of a topological space X. We denote by $\mathrm{cl}_X D$ the closure of D in X and by $\mathrm{int}_X D$ the interior of D in X.

Let X, Y be two nonempty sets and $F : X \to 2^Y$ a mapping. Then the mapping $F^* : Y \to 2^X$ is defined by $F^*(y) := X \setminus F^{-1}(y)$ for each $y \in Y$. Let X be a nonempty set and Y a topological space. A mapping $G : X \to 2^Y$ is said to be transfer compactly closed (resp., open) valued if for each $x \in X$ and for each compact set $K \subset Y$, $y \notin G(x) \cap K$ (resp., $y \in G(x) \cap K$) implies that there exists $x' \in X$ such that $y \notin \mathrm{cl}_K(G(x') \cap K)$ (resp., $y \in \mathrm{int}_K(G(x') \cap K)$)[10, p1058]. Clearly, each open (resp., closed) valued mapping is transfer open (resp., closed) valued and is also compactly open (resp., closed) valued. Each transfer open (resp., closed) valued mapping and each compactly open (resp., closed) valued mapping is transfer compactly open (resp., closed) valued, but the inverse is not true in general.

Following Ben-El-Mechaiekh et al.[11], an L-convexity structure on a topological space X is given by a mapping $\Gamma : \mathcal{F}(X) \to 2^X$ satisfying the following condition: for each $A \in \mathcal{F}(X)$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $B \in \mathcal{F}(A)$ with $|B| = J + 1$, implies $\phi_A(\Delta_J) \subset \Gamma(B)$, where Δ_J denotes the face of Δ_n corresponding $B \in \mathcal{F}(A)$. The pair (X, Γ) is then called an L-convex space. A set $D \subset X$ is said to be L-convex if for each $A \in \mathcal{F}(D)$, $\Gamma(A) \subset D$. We denote by $\mathcal{L}(X)$ the family of all nonempty L-convex subsets of X.

Let X be a nonempty set and (Y, Γ) be an L-convex space. A mapping $G : X \to 2^Y$ is said to be a GLKKM mapping if for each $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exists $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ such that for any nonempty subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$, we have $\Gamma(\{y_{i_j} : j = 1, \ldots, k\}) \subset \bigcup_{j=1}^{k} G(x_{i_j})$.

When (Y, Γ) is a hyperconvex space, H-space, G-convex space or G-H-convex space, the above definition was given by Kirk et al.[5], Chang and Ma[12], Ding[10], Tan[13] and Verma[14], respectively.

The following definition is the improving version of Definition 3.1 of Ding[15].

Definition 1.1 *Let X be a nonempty set, (Y, Γ) an L-convex space and $\gamma \in R$ a real number. A function $g : X \times Y \to \bar{R} := R \cup \{\pm\infty\}$ is said to be generalized γ-L-diagonally quasiconcave* *(resp., quasiconvex) in x if for each $\{x_1, \ldots, x_n\} \in \mathcal{F}(X)$, there exists $\{y_1, \ldots, y_n\} \in \mathcal{F}(Y)$ such that for each nonempty subset $\{y_{i_1}, \ldots, y_{i_k}\} \subset \{y_1, \ldots, y_n\}$ and for each $y \in \Gamma(\{y_{i_1}, \ldots, y_{i_k}\})$, $\min_{1 \le j \le k} g(x_{i_j}, y) \le \gamma$ (resp., $\max_{1 \le j \le k} g(x_{i_j}, y) \ge \gamma$).*

Remark 1.1 In Definition 3.1 of Ding[15], only the generalized γ-L-diagonally quasiconcave was defined. Obviously, the generalized γ-L-diagonally quasiconcave is the generalization of the generalized γ-H-diagonally quasiconcave in Ding and Xia[7].

Clearly, we have the following lemma.

Lemma 1.1 *Let X be a nonempty set, (Y, Γ) an L-convex space and $\gamma \in R$ a real number. Then a function $g : X \times Y \to \bar{R}$ is generalized γ-L-diagonally quasiconcave (resp., generalized γ-L-diagonally quasiconvex) in x if and only if the mapping $G : X \to 2^Y$ defined by $G(x) := \{y \in Y : g(x, y) \le \gamma\}$ (resp., $G(x) := \{y \in Y : g(x, y) \ge \gamma\}$) for each $x \in X$ is a GLKKM mapping.*

Remark 1.2 (1) Lemma 1.1 generalizes Lemma 4.1 of Ding and Xia[7] and Lemma 4.1 of Ding[16] from H-spaces to L-convex spaces under weaker assumptions of g and G.

(2) Note that a GLKKM mapping is the generalization of a GMKKM mapping. By Lemma 1.1 here and Lemma 2.7 of Kirk et al.[5], it is clear that the generalized γ-L-diagonally quasiconcave (resp., generalized γ-L-diagonally quasiconvex) is a generalization of the hyper γ-generalized quasiconcave (resp., hyper γ-generalized quasiconvex) of Kirk et al.[5].

Let (M, d) be a metric space. We denote by μ the usual Kuratowksi measure of noncompactness on M. For any nonempty bounded subset A of M, its closed ball hull $\mathrm{co}(A)$ is defined by $\mathrm{co}(A) := \bigcap\{B \subset M : B$ is a closed ball containing $A\}$. A subset $A \subset M$ is called admissible if $A = \mathrm{co}(A)$. A is said to be finitely metrically closed if for each $F \in \mathcal{F}(M)$, $\mathrm{co}(F) \cap A$ is closed. It is obvious that A is finitely metrically closed if A is closed in M.

In [9], we introduced the following definition.

Definition 1.2[9] *(M, d, Γ) is said to be an L-convex metric space if (M, d) is a metric space and (M, Γ) is an L-convex space such that $\Gamma(A) \subset \mathrm{co}(A)$ for each $A \in \mathcal{F}(M)$.*

Clearly, L-convex metric spaces include hyperconvex spaces of Aronszajn et al.[1−6], H-metric spaces of Ding and Xia[7] and G-convex metric spaces of Meng et al.[8], but the inverse is not true in general. From Definition 1.2, we have the following facts: (a) $\Gamma(\{x\}) = \{x\}$ for each $x \in M$; (b) For each $A \in \mathcal{F}(M)$, $\mathrm{co}(A)$ is L-convex.

The following result is Theorem 2.1 of Wen[9].

Lemma 1.2[9] *Let X be a nonempty set, (M, d, Γ) an L-convex metric space and $T : X \to 2^M \setminus \{\emptyset\}$ a mapping with finitely metrically closed values. Then $\{T(x)\}_{x \in X}$ has the finite intersection property if and only if T is a GLKKM mapping.*

The following result, in which Y need not be a topological space, is the improving version of Lemma 2.1 of Ding[17].

Lemma 1.3 *Let X be a topological space, Y a nonempty set, K a nonempty compact subset* *of X and $G : X \to 2^Y$ a mapping such that $G(x) \neq \emptyset$ for each $x \in K$. Then the following conditions are equivalent:*

(a) G has the compactly local intersection property;

(b) For each $y \in Y$, there exists an open subset O_y of X (which may be empty) such that $O_y \cap K \subset G^{-1}(y)$ and $K = \bigcup_{y\in Y}(O_y \cap K)$;

(c) There exists a mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open or empty in X, $F^{-1}(y) \cap K \subset G^{-1}(y)$, and $K = \bigcup_{y\in Y}(F^{-1}(y) \cap K)$;

(d) For each $x \in K$, there exists $y \in Y$ such that $x \in \mathrm{cint}G^{-1}(y) \cap K$ and

$$K = \bigcup_{y\in Y}(\mathrm{cint}G^{-1}(y) \cap K) = \bigcup_{y\in Y}(G^{-1}(y) \cap K);$$

(e) G^{-1} is transfer compactly open valued on X.

The following definition is the improving version of Definition 4.1 of Ding[18].

Definition 1.3 *Let X be a nonempty set, Y a topological space and $\gamma \in R$ a real number. A function $g : X \times Y \to \bar{R}$ is said to be γ-transfer compactly lower semicontinuous (in short, γ-t.c.l.s.c.) (resp., γ-transfer compactly upper semicontinuous (in short, γ-t.c.u.s.c.)) in y if for each nonempty compactly subset K of Y and for each $x \in X$ and $y \in K$, $g(x, y) > \gamma$ (resp., $g(x, y) < \gamma$) implies that there exist $x' \in X$ and a relatively open neighborhood $\mathcal{N}(y)$ of y in K such that $g(x', z) > \gamma$ (resp., $g(x', z) < \gamma$) for all $z \in \mathcal{N}(y)$.*

Remark 1.3 In Definition 4.1 of Ding[18], X was assumed to be a topological space and only the γ-transfer compactly lower semicontinuous was defined. Obviously, the upper (resp., lower) semicontinuous, transfer upper (resp., lower) semicontinuous and transfer compactly upper (resp., lower) semicontinuous are special cases of γ-transfer compactly upper (resp., lower) semicontinuous, and the W-lower semicontinuous (see Definition 1.5(4) of Zhang[19] and Definition (4) of Wu[20, p285]) is also a special case of γ-transfer compactly lower semicontinuous. Therefore, Definition 1.3 unifies and generalizes Definition 2.6 of Kirk et al.[5], Definition 4.1 of Ding[18], Definition 1.5(4) of Zhang[19], Definition (4) of Wu[20, p285] and Definition 8 of Tian[21].

The following lemma is obvious.

Lemma 1.4 *Let X be a nonempty set, Y a topological space and $\gamma \in R$ a real number. Then a function $g : X \times Y \to \bar{R}$ is γ-t.c.l.s.c. (resp., γ-t.c.u.s.c.) in y if and only if the mapping $G : X \to 2^Y$ defined by $G(x) := \{y \in Y : g(x, y) \leq \gamma\}$ (resp., $G(x) := \{y \in Y : g(x, y) \geq \gamma\}$) for each $x \in X$ is transfer compactly closed valued.*

2. Main results

Theorem 2.1 *Let X be a nonempty set and (M, d, Γ) a complete L-convex metric space. Suppose $G : X \to 2^M$ is a GLKKM mapping with transfer compactly closed values and $\inf_{x\in X} \mu(G(x)) = 0$. Then $\bigcap_{x\in X} G(x)$ is nonempty and compact.*

Proof Define the mapping $\mathrm{cl}G : X \to 2^M$ by $\mathrm{cl}G(x) := \mathrm{cl}_M G(x)$ for each $x \in X$. Then $\mathrm{cl}G$ is closed valued, and hence cl$G$ is finitely metrically closed valued. Since G is a GLKKM mapping, cl$G$ is also a GLKKM mapping. By virtue of Lemma 1.2, $\{\mathrm{cl}G(x)\}_{x\in X}$ has the finite intersection property. Note that M is complete and $\inf_{x\in X}\mu(\mathrm{cl}G(x)) = \inf_{x\in X}\mu(G(x)) = 0$. In virtue of Lemma 4.1 of Kirk et al.[5], $\bigcap_{x\in X}\mathrm{cl}G(x)$ is nonempty and compact.

Now, define a mapping $F : X \to 2^M$ by $F(x) := G(x) \cap \bigcap_{y\in X}\mathrm{cl}G(y)$ for each $x \in X$. Since G is transfer compactly closed valued and $\bigcap_{y\in X}\mathrm{cl}G(y)$ is nonempty and compact, F is transfer closed valued. By Lemma 2.4 of Kirk et al.[5], we have $\bigcap_{x\in X}F(x) = \bigcap_{x\in X}\mathrm{cl}F(x)$. Hence,

$$\begin{aligned}\bigcap_{x\in X}G(x) &= (\bigcap_{x\in X}G(x)) \cap (\bigcap_{x\in X}\mathrm{cl}G(x))\\ &= \bigcap_{x\in X}(G(x)\cap\bigcap_{y\in X}\mathrm{cl}G(y))\\ &= \bigcap_{x\in X}F(x) = \bigcap_{x\in X}\mathrm{cl}F(x)\\ &= \bigcap_{x\in X}\mathrm{cl}(G(x)\cap\bigcap_{y\in X}\mathrm{cl}G(y))\\ &= \bigcap_{x\in X}\mathrm{cl}G(x).\end{aligned}$$

Therefore, $\bigcap_{x\in X}G(x)$ is nonempty and compact.

Remark 2.1 Note that a metric space (M, d) is complete if (M, d) is hyperconvex by Proposition 1 of Khamsi[2]. Suppose there exists $x_0 \in X$ such that $G(x_0)$ is compact. Then $\mu(G(x_0)) = 0$, and hence $\inf_{x\in X}\mu(G(x)) = 0$, certainly. G is transfer compactly closed valued if G is closed valued or transfer closed valued. G is a GLKKM mapping if G is an MKKM mapping. Hence, Theorem 2.1 unifies, improves and generalizes Theorem 4 of Khamsi[2], the KKM theorem of Park[4] and Corollary 2.6 of Kirk et al.[5] from hyperconvex spaces to L-convex metric spaces under weaker assumptions of G. Meanwhile, Theorem 2.1 replaces conclusions of all theorems cited above that $\bigcap_{x\in X}G(x)$ is nonempty by the stronger conclusion that $\bigcap_{x\in X}G(x)$ is nonempty and compact. In addition, the proof method of Theorem 2.1 is different from that of corresponding theorems in the references cited above.

Theorem 2.2 *Let $\gamma \in R$ be a real number, X be a nonempty set and (M, d, Γ) a complete L-convex metric space. Suppose $g : X \times M \to \bar{R}$ such that*

(1) $\inf_{x\in X}\mu(\{y \in M : g(x, y) \le \gamma\}) = 0$ (resp., $\inf_{x\in X}\mu(\{y \in M : g(x, y) \ge \gamma\}) = 0$);

(2) g is generalized γ-L-diagonally quasiconcave (reap., quasiconvex) in x;

(3) $g(x, y)$ is γ-t.c.l.s.c. (resp., γ-t.c.u.s.c.) in y.

Then $\{y \in M : g(x, y) \le \gamma$ for all $x \in X\}$ (resp., $\{y \in M : g(x, y) \ge \gamma$ for all $x \in X\}$) is nonempty and compact.

Proof We only prove the conclusion for the case that $\{y \in M : g(x, y) \le \gamma$ for all $x \in X\}$ is nonempty and compact. The other case can be proved similarly.

Define a mapping $G : X \to 2^M$ by $G(x) := \{y \in M : g(x, y) \le \gamma\}$ for each $x \in X$. Then, by

(1), $\inf_{x\in X}\mu(G(x))=0$. By (2) and Lemma 1.1, G is a GLKKM mapping. By (3) and Lemma 1.4, G is transfer compactly closed valued. Therefore, in virtue of Theorem 2.1, $\bigcap_{x\in M}G(x)$ is nonempty and compact, i.e., $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in M\}$ is nonempty and compact.

Remark 2.2 Since hyperconvex spaces and H-metric spaces are special cases of L-convex metric spaces, the generalized γ-L-diagonally quasiconcave (resp., quasiconvex) is the generalization of the generalized γ-H-diagonally quasiconcave (resp., quasiconvex) and the generalized γ-L-diagonally quasiconcave (resp., generalized γ-L-diagonally quasiconvex) is a generalization of the hyper γ-generalized quasiconcave (resp., hyper γ-generalized quasiconvex). Then, Theorem 2.2 improves and generalizes Theorem 2.8 of Kirk et al.[5], Theorem 4.1 of Ding and Xia[7], Theorem 4 of Liu[22] and Theorem 2.11.15 of Yuan[23] in several aspects, and replaces conclusions of all theorems cited above that $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in X\}$ (resp., $\{y\in M: g(x,y)\ge\gamma \text{ for all } x\in X\}$) is nonempty by the stronger conclusion that $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in X\}$ (resp., $\{y\in M: g(x,y)\ge\gamma \text{ for all } x\in X\}$) is nonempty and compact.

Corollary 2.1 *Let $\gamma\in R$ be a real number, (M,d,Γ) a complete L-convex metric space and X a nonempty L-convex subset of M. Suppose $g: X\times M\to\bar{R}$ such that*

(1) $\inf_{x\in X}\mu(\{y\in M: g(x,y)\le\gamma\})=0$ (resp., $\inf_{x\in X}\mu(\{y\in M: g(x,y)\ge\gamma\})=0$);

(2) For each $y\in M$, $\{x\in X: g(x,y)>\gamma\}$ (resp., $\{x\in X: g(x,y)<\gamma\}$) $=0$ or $\in\mathcal{L}(M)$;

(3) For each $x\in X$, $g(x,x)\le\gamma$ (resp., $g(x,x)\ge\gamma$);

(4) $g(x,y)$ is γ-t.c.l.s.c. (resp., γ-t.c.u.s.c.) in y.

Then $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in X\}$ (resp., $\{y\in M: g(x,y)\ge\gamma \text{ for all } x\in X\}$) is nonempty and compact.

Proof We only prove the conclusion for the case that $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in X\}$ is nonempty and compact.

Define a mapping $G: X\to 2^M$ by $G(x):=\{y\in M: g(x,y)\le\gamma\}$ for each $x\in X$. We claim that G is a GLKKM mapping. Otherwise, there exists $\{x_1,\dots,x_n\}\in\mathcal{F}(X)$, for each $\{y_1,\dots,y_n\}\in\mathcal{F}(M)$, there exist a nonempty subset $\{y_{i_1},\dots,y_{i_k}\}$ of $\{y_1,\dots,y_n\}$ and $y\in\Gamma(\{y_{i_1},\dots,y_{i_k}\})$ such that $y\notin\bigcup_{j=1}^k G(x_{i_j})$. Especially, for $\{x_1,\dots,x_n\}\in\mathcal{F}(X)\subset\mathcal{F}(M)$, there exist a nonempty subset $\{x_{i_1},\dots,x_{i_k}\}$ of $\{x_1,\dots,x_n\}$ and $y_0\in\Gamma(\{x_{i_1},\dots,x_{i_k}\})$ such that $y_0\notin\bigcup_{j=1}^k G(x_{i_j})$, which results in that for all $x_{i_j}\in\{x_{i_1},\dots,x_{i_k}\}$, $g(x_{i_j},y_0)>\gamma$. By (2), we have $\{x_{i_1},\dots,x_{i_k}\}\subset\{x\in X: g(x,y_0)>\gamma\}\in\mathcal{L}(M)$. Thus $y_0\in\Gamma(\{x_{i_1},\dots,x_{i_k}\})\subset\{x\in X: g(x,y_0)>\gamma\}$. Hence, $g(y_0,y_0)>\gamma$, which contradicts (3). Therefore, G is a GLKKM mapping. By Lemma 1.1, g is generalized γ-L-diagonally quasiconcave in x. By (1) and (4), in virtue of Theorem 2.2, $\{y\in M: g(x,y)\le\gamma \text{ for all } x\in X\}$ is nonempty and compact.

Theorem 2.3 *Let (M,d,Γ) be a complete L-convex metric space, X be a nonempty L-convex subset of M and $A\subset X\times M$ a nonempty subset such that*

(1) $\inf_{x\in X}\mu(\{y\in M: (x,y)\in A\})=0$;

(2) For each $y\in M$, $\{x\in X: (x,y)\notin A\}=\emptyset$ or $\in\mathcal{L}(M)$;

(3) For each $x\in X$, $(x,x)\in A$;

(4) The mapping $G : X \to 2^M$ defined by $G(x) := \{y \in M : (x, y) \in A\}$ for each $x \in X$ is transfer compactly closed valued;
Then $\{y \in M : X \times \{y\} \subset A\}$ is nonempty and compact.

Proof Define a function $g : X \times M \to R$ by

$$g(x,y) := \begin{cases} 1, & \text{if } (x,y) \in A, \\ 0, & \text{if } (x,y) \notin A, \end{cases}$$

for each $(x, y) \in X \times M$. Then for each $x \in X$, $G(x) := \{y \in M : (x, y) \in A\} = \{y \in M : g(x, y) \geq 1\}$. Moreover, by (1), $\inf_{x\in X} \mu(G(x)) = 0$. By (2), for each $y \in M$, $\{x \in X : g(x, y) < 1\} = \emptyset$ or $\in \mathcal{L}(M)$. By (3), for each $x \in X$, $g(x, x) \geq 1$. By (4), $G : X \to 2^M$ defined by $G(x) = \{y \in M : g(x, y) \geq 1\}$ for each $x \in X$ is transfer compactly closed valued. In virtue of Lemma 1.4, $g(x, y)$ is 1-t.c.u.s.c. in y. Therefore, by virtue of Corollary 2.1, $\{y \in M : g(x, y) \geq 1$ for all $x \in X\}$ is nonempty and compact, which implies that $\{y \in M : X \times \{y\} \subset A\}$ is nonempty and compact.

Remark 2.3 If $X = M$ is a compact admissible subset of a hyperconvex space, M is a complete L-convex metric space and the condition (1) is satisfied trivially. If for each $y \in M$, $\{x \in X : (x, y) \notin A\}$ is admissible or sub-admissible, of course, the condition (2) is satisfied trivially. If G is closed valued or transfer closed valued, the condition (4) is certainly satisfied. Therefore, Theorem 2.3 improves and generalizes Theorem 3.2 of Kirk et al.[5], Theorem 3.2 of Wen[6] and Theorem 3 of Chen and Shen[24] in several aspects and strengthens all conclusions of theorems cited above.

Theorem 2.4 *Let (M, d, Γ) be a complete L-convex metric space, X be a nonempty L-convex subset of M and $G : X \to 2^M$ a mapping such that*

(1) $\inf_{x\in X} \mu(G(x)) = 0$;

(2) For each $y \in M$, $G^*(y) = \emptyset$ or $\in \mathcal{L}(M)$;

(3) For each $x \in X$, $x \in G(x)$;

(4) G is transfer compactly closed valued.

Then $\bigcap_{x\in X} G(x)$ is nonempty and compact.

Proof Let $A := \{(x, y) \in X \times M : y \in G(x)\}$. Then $G(x) = \{y \in M : (x, y) \in A\}$ for each $x \in X$, and $G^*(y) = X \setminus G^{-1}(y) = \{x \in X : y \notin G(x)\} = \{x \in M : (x, y) \notin A\}$ for each $y \in M$. Hence, by (1), $\inf_{x\in X} \mu(\{y \in M : (x, y) \in A\}) = 0$. By (2), for each $y \in M$, $\{x \in X : (x, y) \notin A\} = \emptyset$ or $\in \mathcal{L}(M)$. By (3), for each $x \in X$, $(x, x) \in A$. By (4), the mapping $G : X \to 2^M$ defined by $G(x) := \{y \in M : (x, y) \in A\}$ for each $x \in X$ is transfer compactly closed valued. In virtue of Theorem 2.3, $\{y \in M : X \times \{y\} \subset A\}$ is nonempty and compact, which implies that $\bigcap_{x\in X} G(x)$ is nonempty and compact.

Remark 2.4 If $X = M$ is a compact admissible subset of a hyperconvex space, G^* is admissible valued and G is closed, then M is a complete L-convex metric space and the conditions (1), (2) and (4) are satisfied trivially. Therefore, Theorem 2.4 improves and generalizes Theorem 3.3 of

Kirk et al.[5] and Theorem 3.3 of Wen[6] in several aspects and strengthens their conclusions.

Theorem 2.5 *Let (M, d, Γ) be a complete L-convex metric space, X be a nonempty L-convex subset of M and $G : M \to 2^X$ such that*

(1) $\inf_{x\in X} \mu(G^*(x)) = 0$;

(2) For each $y \in M$, $G(y) = \emptyset$ or $\in \mathcal{L}(M)$;

(3) For each $x \in X$, $x \notin G(x)$;

(4) G^{-1} is transfer compactly open valued.

Then $\{x \in M : G(x) = \emptyset\}$ is nonempty and compact.

Proof Let $A := \{(x, y) \in X \times M : x \notin G(y)\}$. Note that for each $x \in X$, $G^*(x) = M \setminus G^{-1}(x) = \{y \in M : x \notin G(y)\} = \{y \in M : (x, y) \in A\}$, and for each $y \in M$, $G(y) = \{x \in X : x \in G(y)\} = \{x \in X : (x, y) \notin A\}$. Then, by (1), $\inf_{x\in X} \mu(\{y \in M : (x, y) \in A\}) = 0$. By (2), for each $y \in M$, $\{x \in X : (x, y) \notin A\} = \emptyset$ or $\in \mathcal{L}(M)$. By (3), for each $x \in X$, $(x, x) \in A$. By (4), the mapping G^* defined by $G^*(x) := \{y \in M : (x, y) \in A\}$ for each $x \in X$ is transfer compactly closed valued. In virtue of Theorem 2.3, $\{y \in M : X \times \{y\} \subset A\}$ is nonempty and compact. Therefore, $\{x \in M : G(x) = \emptyset\}$ is nonempty and compact.

Remark 2.5 As we have noted in remarks above, Theorem 2.5 improves and generalizes Theorem 3.4 of Kirk et al.[5] in several aspects and strengthens its conclusion. In addition, the proof method of Theorem 2.5 is different from that of corresponding theorems in references.

As an immediate consequence of Theorem 2.5, we have the following Fan-Browder fixed point theorem in noncompact L-convex metric spaces.

Theorem 2.6 *Let (M, d, Γ) be a complete L-convex metric space and $G : M \to 2^M \setminus \{\emptyset\}$ such that*

(1) $\inf_{x\in M} \mu(G^*(x)) = 0$;

(2) For each $y \in M$, $G(x) \in \mathcal{L}(M)$;

(3) G satisfies one of the conditions (a)$\sim$(e) in Lemma 1.3.

Then there exists $x_0 \in M$ such that $x_0 \in G(x_0)$.

Remark 2.6 Theorem 2.6 unifies, improves and generalizes Theorem 3 of Park[4], Theorem 3.1 of Kirk[5], Theorem 3.1 of Wen[6], Lemma 2.2 of Zhang[19], Corollarys 2 and 3 of Chen and Shen[24] and Theorem 8 of Park[25].

References

[1] ARONSZAJN N and PANITCHPAKDI P. *Extension of uniformly continuous transformation and hyperconvex metric space* [J]. Pacific J. Math., 1956, **6**: 405–439.

[2] KHAMSI M A. *KKM and Ky Fan theorems in hyperconvex metric spaces* [J]. J. Math. Anal. Appl., 1996, **204**(1): 298–306.

[3] YUAN Xianzhi. *The characterization of generalized metric KKM mappings with open values in hyperconvex metric spaces and some applications* [J]. J. Math. Anal. Appl., 1999, **235**(1): 315–325.

[4] PARK S. *Fixed point theorems in hyperconvex metric spaces* [J]. Nonlinear Anal., 1999, **37**(4): 467–472.

[5] KIRK W A, SIMS B, YUAN Xianzhi. *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications* [J]. Nonlinear Anal., 2000, **39**(5): 611–627.

[6] WEN Kaiting. *A Browder fixed point theorem in noncompact hyperconvex metric spaces and its applications to coincidence problems* [J]. Adv. Math. (China), 2005, **34**(2): 208–212.
[7] DING Xieping, XIA Fuquan. *Generalized H-KKM type theorems in H-metric spaces with application* [J]. Appl. Math. Mech. (English Ed.), 2001, **22**(10): 1140–1148.
[8] MENG Li, SHEN Zifei, CHENG Xiaoli. *A generalized KKM type theorem for G-convex metric spaces and its applications* [J]. Acta Anal. Funct. Appl., 2005, **7**(3): 273–279. (in Chinese)
[9] WEN Kaiting. *GLKKM theorems in L-convex metric spaces with application* [J]. Acta Anal. Funct. Appl., 2008, **10**(2): 109–115.
[10] DING Xieping. *Coincidence theorems and equilibria of generalized games* [J]. Indian J. Pure Appl. Math., 1996, **27**(11): 1057–1071.
[11] BEN-EL-MECHAIEKH H, CHEBBI S, FLORNZANO M. et al. *Abstract convexity and fixed points* [J]. J. Math. Anal. Appl., 1998, **222**(1): 138–150.
[12] CHANG S S, MA Yihai. *Generalized KKM theorem on H-space with applications* [J]. J. Math. Anal. Appl., 1992, **163**(2): 406–421.
[13] TAN, K K. *G-KKM theorem, minimax inequalities and saddle points* [J]. Nonlinear Anal., 1997, **30**(7): 4151–4160.
[14] VERMA R U. *G-H-KKM type theorems and their applications to a new class of minimax inequalities* [J]. Comput. Math. Appl., 1999, **37**(8): 45–48.
[15] DING Xieping. *Generalized L-KKM type theorems in L-convex spaces with applications* [J]. Comput. Math. Appl., 2002, **43**(10-11): 1249–1256.
[16] DING Xieping. *Fixed points, minimax inequalities and equilibria of noncompact abstract economies* [J]. Taiwanese J. Math., 1998, **2**(1): 25–55.
[17] DING Xieping. *Generalized variational inequalities and equilibrium problems in generalized convex spaces* [J]. Comput. Math. Appl., 1999, **38**(7-8): 189–197.
[18] DING Xieping. *New H-KKM theorems and their applications to geometric property, coincidence theorems, minimax inequality and maximal elements* [J]. Indian J. Pure Appl. Math., 1995, **26**(1): 1–19.
[19] ZHANG Huili. *Some nonlinear problems in hyperconvex metric spaces* [J]. J. Appl. Anal., 2003, **9**(2): 225–235.
[20] WU Xian. *Existence theorems for maximal elements in H-spaces with applications on the minimax inequalities and equilibrium of games* [J]. J. Appl. Anal., 2000, **6**(2): 283–293.
[21] TIAN Guoqiang. *Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity* [J]. J. Math. Anal. Appl., 1992, **170**(2): 457–471.
[22] LIU Xuewen. *Non compact versious of the KKM principle in complete H-metric spaces and some of its applications* [J]. Xi'nan Shifan Daxue Xuebao Ziran Kexue Ban, 2003, **28**(2): 159–161. (in Chinese)
[23] YUAN Xianzhi. *KKM theory and applications in nonlinear analysis* [M]. New York: Marcel Dekker Inc, 1999.
[24] CHEN Fengjuan, SHEN Zifei. *Continuous selection theorems and coincidence theorems on hyperconvex spaces* [J]. Adv. Math. (China), 2005, **34**(5): 614–618.
[25] PARK S. *Fixed point theorems in locally G-convex spaces* [J]. Nonlinear Anal., 2002, **48**(6): 869–879.