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The Total Stiefel-Whitney Classes of Vector Bundles on $CP(n)\times CP(m)$

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Abstract The possible form of the total Stiefel-Whitney classes of vector bundles on $CP(n)\times CP(m)$ is determined in this paper.

Keywords vector bundle; total Stiefel-Whitney class; Wu formula.

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1. Introduction

In 1962, Steenrod raised to Conner the following question

Given a smooth closed manifold F, does there exist a non-trivial involution (M,T) on a smooth closed manifold M such that the fixed set of T is F? Can one determine all involutions (M,T) for a special F?

For the sake of convenience, people think about determining the bordism classes of involutions (M,T) which has F as its fixed set. When F is a sphere, a projective space, or the disjoint union of them, and the case F is a Dold manifold, there are some results in [1]–[5]. But there are few results for the case that F is the product of some spaces. Stong, Weiss and Saiers discussed in [6], [7] the case that F is the product of two real projective spaces.

From [8], one knows that the bordism classes of involutions with F as its fixed set is determined by the bordism classes of normal bundles on F. For this reason, to determine involutions which has $F=CP(n)\times CP(m)$ as its fixed set, one must know the possible form of the total Stiefel-Whitney classes of vector bundles on it.

The main result of this paper is the following theorem.

Theorem *The total Stiefel-Whitney class of a vector bundle ξ on $CP(n)\times CP(m)$ must have the form*

$$W(\xi)=(1+z_1)^a(1+z_2)^b(1+z_1+z_2)^c(1+z_1^iz_2^{2^{s-1}-i})^{\varepsilon},$$

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where $z_1 \in H^2(CP(n);Z_2), z_2 \in H^2(CP(m);Z_2)$ are generators, a, b, c are non-negative integers. $\varepsilon = 0$ or 1, and when $\varepsilon = 1$, we must have

$$\begin{cases} i = 2^t(2p+1), & t \geq 0, \\ n = 2^t(2p+1)+x, & 0 \leq x < 2^t, \\ m = 2^{s-1} - 2^t(2p+1)+y, & 0 \leq y < 2^t. \end{cases}$$

Throughout this paper, $CP(n)$ denotes the n dimensional complex projective space, $c_i(\xi)$ is the ith Chern class of the complex vector bundle ξ, $w_i(\xi)$ is the ith Stiefel-Whitney class of the real vector bundle ξ, and $W(\xi)$ denotes the total Stiefel-Whitney class of the real vector bundle ξ.

All vector bundles in this paper are real vector bundles unless there is a special announcement.

2. The total Stiefel-Whitney classes of vector bundles on $CP(n)\times CP(m)$

Let

$$H^*(CP(n)\times CP(m);Z) = Z[z_1]/z_1^{n+1} \otimes Z[z_2]/z_2^{m+1},$$

where $z_1 \in H^2(CP(n);Z), z_2 \in H^2(CP(m);Z)$ are generators. For convenience, we denote generators of $H^2(CP(n);Z_2)$, $H^2(CP(m);Z_2)$ also by z_1, z_2.

Let γ_1 denote the canonical complex line bundle over $CP(n)$, γ_2 denote the canonical complex line bundle over $CP(m)$. $p_1 : CP(n)\times CP(m) \to CP(n)$, $p_2 : CP(n)\times CP(m) \to CP(m)$ are projections. Then pullbacks $p_1^*(\gamma_1)$ and $p_2^*(\gamma_2)$ are complex line bundles over $CP(n)\times CP(m)$. Thereby, $p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)$ is also a complex line bundle over $CP(n)\times CP(m)$.

Lemma 1 *The first Chern class of the bundle $p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)$ is $c_1p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)) = z_1 + z_2$.*

Proof We define a map $i_1 : CP(n) \to CP(n)\times CP(m)$ by $i_1(x) = (x, pt_2)$, for every $x \in CP(n)$; define a map $i_2 : CP(m) \to CP(n)\times CP(m)$ by $i_2(x) = (pt_1, x)$, for every $x \in CP(m)$, where $pt_1 \in CP(n), pt_2 \in CP(m)$ are fixed. So we have

$$p_1 i_1 : CP(n) \to CP(n) \text{ is the identity on } CP(n),$$

$$p_2 i_2 : CP(m) \to CP(m) \text{ is the identity on } CP(m).$$

Therefore, we have

$$(p_1 i_1)^*(\gamma_1) = i_1^* p_1^*(\gamma_1) = \gamma_1, \tag{1}$$

and

$$(p_2 i_2)^*(\gamma_2) = i_2^* p_2^*(\gamma_2) = \gamma_2. \tag{2}$$

From (1), we have

$$i_1^*(c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2))) = c_1(i_1^*p_1^*(\gamma_1)\otimes i_1^*p_2^*(\gamma_2)) = c_1(\gamma_1) = z_1. \tag{3}$$

For the same reason, from (2), we may get

$$i_2^*(c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2))) = z_2.$$

Let $c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2))=\varepsilon_1 z_1+\varepsilon_2 z_2$. Then we have

$$i_1^*(c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)))=i_1^*(\varepsilon_1 z_1+\varepsilon_2 z_2)=\varepsilon_1 z_1,$$

from (3) we know that $\varepsilon_1=1$. For the same reason we may get $\varepsilon_2=1$. So we have $c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2))=z_1+z_2$.

Lemma 2 *There exists a 2-dimensional vector bundle η over $CP(n)\times CP(m)$ such that $W(\eta)=1+z_1+z_2$.*

Proof Let η be the realification of the complex bundle $p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)$. Then $w_2(\eta)=c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2)) \bmod 2$. From Lemma 1, one knows that $c_1(p_1^*(\gamma_1)\otimes p_2^*(\gamma_2))=z_1+z_2$, so we have $w_2(\eta)=z_1+z_2$. Therefore, we have $W(\eta)=1+z_1+z_2$.

Lemma 3[9] *Let the total Stiefel-Whitney class of a vector bundle ξ be*

$$W(\xi)=1+w_{2^s}+\textit{higher terms}.$$

Then $Sq^i w_{2^s}=0, 0<i<2^{s-1}$.

Theorem *The total Stiefel-Whitney class of a vector bundle ξ on $CP(n)\times CP(m)$ must have the form*

$$W(\xi)=(1+z_1)^a(1+z_2)^b(1+z_1+z_2)^c(1+z_1^i z_2^{2^{s-1}-i})^{\varepsilon},$$

where $z_1\in H^2(CP(n);Z_2)$, $z_2\in H^2(CP(m);Z_2)$ are generators, a,b,c are non-negative integers. $\varepsilon=0$ or 1, and when $\varepsilon=1$, we must have

$$\begin{cases} i=2^t(2p+1), & t\geq 0,\\ n=2^t(2p+1)+x, & 0\leq x<2^t,\\ m=2^{s-1}-2^t(2p+1)+y, & 0\leq y<2^t.\end{cases}$$

Proof Let $p_1^*(\gamma_1), p_2^*(\gamma_2)$ as former. For convenience, denote their realification also by $p_1^*(\gamma_1), p_2^*(\gamma_2)$. Therefore we have $W(p^*(\gamma_1))=1+z_1, W(p^*(\gamma_2))=1+z_2$.

Let $W(\xi)=1+\varepsilon_1 z_1+\varepsilon_2 z_2+\text{higher terms}$. If $\varepsilon_1=1,\varepsilon_2=0$, we may have $W(\xi-p^*(\gamma_1))=1+w_4+\text{higher terms}$; if $\varepsilon_1=0,\varepsilon_2=1$, we may have $W(\xi-p^*(\gamma_2))=1+w_4+\text{higher terms}$; if $\varepsilon_1=1,\varepsilon_2=1$, we may have $W(\xi-\eta)=1+w_4+\text{higher terms}$, where η is the vector bundle in Lemma 2.

Let $w_4=\varepsilon_1 z_1^2+\varepsilon_2 z_1 z_2+\varepsilon_3 z_2^2$. Then from $W(2p^*(\gamma_1))=1+z_1^2, W(2p^*(\gamma_2))=1+z_2^2$, and

$$W(p^*(\gamma_1)+p^*(\gamma_2)-\eta)=\frac{(1+z_1)(1+z_2)}{1+z_1+z_2}=1+z_1z_2+\text{higher terms}$$

(note 1: From $\frac{(1+z_1)(1+z_2)}{1+z_1+z_2}=\frac{(1+z_1)(1+z_2)(1+z_1+z_2)^{2^N}}{1+z_1+z_2}=(1+z_1)(1+z_2)(1+z_1+z_2)^{2^N-1}$, one knows that it must be the total Stiefel-Whitney class of some vector bundle, where N is sufficiently large.) We may obtain a vector bundle ξ_2, such that $W(\xi-\xi_2)=1+w_8+\text{higher terms}$. Proceeding inductively, we may suppose that a vector bundle θ which is composed of multiples of $p^*(\gamma_1), p^*(\gamma_2), \eta$ has been found such that

$$W(\xi-\theta)=1+w_{2^s}+\text{higher terms}.$$

(note 2: see also [10], page 94, Problem 8-B.) Since $W(2^{s-1}p^*(\gamma_1)) = 1+z_1^{2^{s-1}}, W(2^{s-1}p^*(\gamma_2)) = 1+z_2^{2^{s-1}}$, and

$$W(2^{s-2}(p^*(\gamma_1)+p^*(\gamma_2)-\eta)) = 1+z_1^{2^{s-2}}z_2^{2^{s-2}} + \text{higher terms},$$

we may also suppose that

$$w_{2^s}(\xi-\theta) = \sum a_i z_1^i z_2^{2^{s-1}-i}, \quad \text{with} \quad i \neq 0, 2^{s-2}, 2^{s-1}.$$

For all values of i such that $a_i \neq 0$, we may suppose that they are all divisible by $2^t(0<2^t<2^{s-2})$ with at least one i being an odd multiple of 2^t.

If a monomial $z_1^h z_2^{2^{s-1}-h}$ occurs in $w_{2^s}(\xi-\theta)$, then we have

$$\begin{aligned} Sq^{2\cdot 2^t}(z_1^h z_2^{2^{s-1}-h}) &= \binom{h}{2^t} z_1^{h+2^t} z_2^{2^{s-1}-h} + \binom{2^{s-1}-h}{2^t} z_1^h z_2^{2^{s-1}-h+2^t} \\ &= z_1^{h+2^t} z_2^{2^{s-1}-h} + z_1^h z_2^{2^{s-1}-h+2^t} \end{aligned}$$

while $h = 2^t(2p+1)$. If h is an even multiple of 2^t, one has $Sq^{2\cdot 2^t}(z_1^h z_2^{2^{s-1}-h}) = 0$. From Lemma 3, one knows

$$Sq^i w_{2^s} = 0, i < 2^{s-1}.$$

Therefore, we have

$$0 = Sq^{2\cdot 2^t} w_{2^s} = \sum_h Sq^{2\cdot 2^t}(z_1^h z_2^{2^{s-1}-h}) = \sum_h (z_1^{h+2^t} z_2^{2^{s-1}-h} + z_1^h z_2^{2^{s-1}-h+2^t}), \tag{4}$$

where h in the right side of the third equal sign are odd multiples of 2^t.

For $h = 2^t(2p+1), h' = 2^t(2q+1)$, we have

$$Sq^{2\cdot 2^t}(z_1^h z_2^{2^{s-1}-h}) = z_1^{h+2^t} z_2^{2^{s-1}-h} + z_1^h z_2^{2^{s-1}-h+2^t},$$

and

$$Sq^{2\cdot 2^t}(z_1^{h'} z_2^{2^{s-1}-h'}) = z_1^{h'+2^t} z_2^{2^{s-1}-h'} + z_1^{h'} z_2^{2^{s-1}-h'+2^t}.$$

Since $h \neq h'+2^t, h' \neq h+2^t$, we know that (4) implies that

$$z_1^{h+2^t} z_2^{2^{s-1}-h} = z_1^h z_2^{2^{s-1}-h+2^t} = 0,$$

for every $h = 2^t(2p+1)$.

So, if $w_{2^s} \neq 0$, then there must be a monomial $z_1^i z_2^{2^{s-1}-i}$, $i = 2^t(2p+1)$, in w_{2^s} such that

$$z_1^{i+2^t} z_2^{2^{s-1}-i} = z_1^i z_2^{2^{s-1}-i+2^t} = 0.$$

Therefore, we get

$$i \leq n, 2^{s-1}-i \leq m; \text{ (since } \ z_1^i z_2^{2^{s-1}-i} \neq 0)$$

$$n < i+2^t, m < 2^{s-1}-i+2^t.$$

Since other monomials in w_{2^s} must have the form $z_1^l z_2^{2^{s-1}-l}$ (l is a multiple of 2^t), if $l > i$, then $l \geq i+2^t > n$, we may have $z_1^l z_2^{2^{s-1}-l} = 0$; if $l < i$, then $2^{s-1}-l > 2^{s-1}-i$, thereby $2^{s-1}-l \geq 2^{s-1}-i+2^t$, we may also have $z_1^l z_2^{2^{s-1}-l} = 0$. So we get

$$w_{2^s} = z_1^i z_2^{2^{s-1}-i}, i = 2^t(2p+1).$$

From the proof of Lemma 3, one knows that $w_{2^s+l}=0$, for $0<l<2^{s-1}$. We may prove that $w_{2^s+l}=0$, for $l\geq 2^{s-1}$. It is sufficient to prove that $z_1^u z_2^v=0$, for $u+v\geq 2^{s-1}+2^{s-2}$. If $u\geq i+2^t$, then $u>n$, so $z_1^u z_2^v=0$; if $u<i+2^t$, then

$$v\geq 2^{s-1}+2^{s-2}-u>2^{s-1}+2^{s-2}-i-2^t\geq 2^{s-1}-i+2^t>m,$$

so $z_1^u z_2^v=0$.

From the above discussion, we may conclude that the total Stiefel-Whitney class of a vector bundle ξ on $CP(n)\times CP(m)$ must have the form

$$W(\xi)=(1+z_1)^a(1+z_2)^b(1+z_1+z_2)^c(1+z_1^i z_2^{2^{s-1}-i})^\varepsilon,$$

where $z_1\in H^2(CP(n);Z_2), z_2\in H^2(CP(m);Z_2)$ are generators, $\varepsilon=0$ or 1, and when $\varepsilon=1$,we must have

$$\begin{cases} i=2^t(2p+1), & t\geq 0,\\ n=2^t(2p+1)+x, & 0\leq x<2^t,\\ m=2^{s-1}-2^t(2p+1)+y, & 0\leq y<2^t.\end{cases}$$

From note 1, one knows that a,b,c are non-negative integers.

Remark We do not know whether there is a vector bundle over $CP(n)\times CP(m)$ whose total Stiefel-Whitney class is $1+z_1^i z_2^{2^{s-1}-i}$.

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