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Iterative Methods for Solving a System of Variational Inclusions Involving (H,η)-Monotone Operators in Banach Spaces

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Abstract In this paper, we introduce and study a new system of variational inclusions involving (H,η)-monotone operators in Banach space. Using the resolvent operator associated with (H,η)-monotone operators, we prove the existence and uniqueness of solutions for this new system of variational inclusions. We also construct a new algorithm for approximating the solution of this system and discuss the convergence of the iterative sequence generated by the algorithm.

Keywords q-uniformly smooth space; (H,η)-monotone operator; Resolvent operator technique; system of variational inclusion; iterative algorithm.

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1. Introduction and preliminaries

Variational inclusions are important generalization of classical variational inequalities and thus, have found wide applications in many fields, including, for example, mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences. For these reasons, various variational inclusions have been intensively studied in recent years. For details, we refer the reader to [1]–[10] and the references therein. Recently, some interesting and important problems related to variational inequalities and complementarity problems have been considered by many authors. Ansari and Yao[11] studied a system of variational inequalities using a fixed-point theorem. Huang and Fang[12] introduced a system of order complementarity problems and established some existence results for these using fixed-point theory.

On the other hand, monotonicity techniques were extended and applied in recent years because of their importance in the theory of variational inequalities, complementarity problems, and variational inclusions. In [4], Huang and Fang introduced a class of generalized monotone operators, maximal η-monotone operators, and defined an associated resolvent operator. Using resolvent operator methods, they developed some iterative algorithms to approximate

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the solution of a class of general variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator. Fang and Huang introduced another class of generalized monotone operators, H-monotone operators, and defined an associated resolvent operator. They also established the Lipschitz continuity of the resolvent operator and studied a class of variational inclusions in Hilbert spaces using the resolvent operator associated with H-monotone operators. In 2005, Fang and Huang further introduced a new class of generalized monotone operators, i.e., (H,η)-monotone operators, which provide a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. They also studied a class of variational inclusions using the resolvent operator associated with an (H,η)-monotone operator.

Motivated and inspired by above works, in this paper, we introduce and study a new system of variational inclusions involving (H,η)-monotone operators in Banach spaces. We prove the existence and uniqueness of solutions for this new system of variational inclusions. We also construct a new algorithm to approximate the solution of this system of variational inclusions and discuss the convergence of iterative sequences generated by the algorithm. The present results improve and extend many known results in the literature. This paper discusses the above mentioned problems in Banach spaces and breaks through the restriction that the space X is Hilbert space. Thus our research for the variational inclusions can be applied to such spaces as L^p and $W^{m,p}(\Omega)(p>1)$.

In what follows, let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q(x): X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q = \|x\|^{q-2}J_2$, for all $x \in X$, and $J_q(x)$ is single-valued if X^* is strictly convex. In the sequel, unless otherwise specified, we always suppose that X is a real Banach space such that $J_q(x)$ is single-valued and H is a Hilbert space. If $X = H$, then J_2 becomes the identity mapping of H.

The modulus of smoothness of X is the function $\rho_X : [0,+\infty) \to [0,+\infty)$ defined by

$$\rho_X(t) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\| - 1) : \|x\| \le 1, \|y\| \le t\Big\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to\infty} \frac{\rho_X(t)}{t} = 0,$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \le ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu[6] proved the following Lemma.

Lemma 1.1[6] *Let X be a real uniformly smooth Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all$x, y \in X$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{1.1}$$

Definition 1.1[10] *Let $T : X \to X^*$ be a single-valued operator. The operator T is said to be*

(1) Monotone if $\forall x, y \in X$,

$$\langle Tx - Ty, x - y\rangle \geq 0;$$

(2) Strictly monotone if T is monotone and $\langle Tx - Ty, x - y\rangle = 0$ if and only if $x = y$;

(3) Strongly monotone if there exists some constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2;$$

(4) Lipschitz continuous if there exists a constant $s > 0$, such that

$$\|Tx - Ty\| \leq s\|x - y\|.$$

Definition 1.2[10] *Let $M : X \to 2^{X^*}$ be a multivalued operator, and $H : X \to X^*$ and $\eta : X \times X \to X$ be single-valued operators. M is said to be:*

(1) Monotone if $\langle x - y, u - v\rangle \geq 0$, $\forall u, v \in X$, $x \in Mu, y \in Mv$;

(2) η-monotone if $\langle x - y, \eta(u - v)\rangle \geq 0$, $\forall u,\ v \in Xx \in Mu,\ y \in Mv$;

(3) Strongly η-monotone if there exists some constant $r > 0$, such that

$$\langle x - y, \eta(u, v)\rangle \geq r\|u - v\|^2, \quad \forall u, v \in X, x \in Mu, y \in Mv.$$

When M is single-valued operator, above formula becomes

$$\langle Mu - Mv, \eta(u, v)\rangle \geq r\|u - v\|^2, \quad \forall u, v \in X;$$

(4) Maximal monotone if M is monotone and has no proper monotone extension in X, i.e., for $\forall u, v_0 \in X$, $x \in Mu$, $\langle x - y_0, u - v_0\rangle \geq 0$ implies $y_0 \in Mv_0$. When X is reflective Banach space, M is maximal monotone if and only if $(J + \lambda M)X = X^*$, for $\forall\lambda > 0$;

(5) Maximal η-monotone if M is η-monotone and has no proper η-monotone extension in X;

(6) H-monotone if M is monotone and $(H + \lambda M)X = X^*$, for all $\lambda > 0$;

(7) (H, η)-monotone if M is η-monotone and $(H + \lambda M)X = X^*$, for all $\lambda > 0$.

Remark 1.1 Maximal η-monotone operators, H-monotone operators, and (H, η)-monotone operators were first introduced by Huang and Fang[12]. Obviously, the class of (H, η)-monotone operators provides a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. For details about these operators, we refer the reader to [5, 10–12] and the references therein.

For our results, we need the following concepts and definitions.

Definition 1.3[10] *Let $\eta : X \times X \to X$ and $H : X \to X^*$ be two single-valued operators, and $M : X \to 2^{X^*}$ be an (H, η)-monotone operator. The resolvent operator $R^{H,\eta}_{M,\lambda} : X^* \to X$ is defined by $R^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u)$.*

In virtue of $(H+\lambda M)X = X^*$, then $\forall u \in X^*$, $R^{H,\eta}_{M,\lambda}$ is a matter of significance, and we can show that it is single-valued in following Theorem 2.1.

Remark 1.2 In [5, Proposition 2.1], Fang and Huang also showed that when H is bounded, coercive, hemi-continuous, and monotone, $M : X \to 2^{X^*}$ is a maximal monotone operator, it follows from Corollary 32.26 of [13] that $H + \lambda M$ is surjective, i.e., $(H+\lambda M)X = X^*$ holds for every $\lambda > 0$, in reflective Banach space.

Definition 1.4[6] *The operator $T : X \to X^*$ is said to be strongly accretive with respect to H if there exists some constant $r > 0$, such that*

$$\langle Tx - Ty, J_q^*(Hx - Hy)\rangle \geq r\|x-y\|^q, \quad \forall x, y \in X,$$

where $J_q^* : X^* \to X^{**}$ is the generalized duality mapping on X^*.

2. (H,η)-monotone operators and resolvent operator technique

To study the iterative algorithm for a system of variational inclusions with (H,η)-monotone operators, now we prove the Lipschitz continuity of the resolvent operator $R^{H,\eta}_{M,\lambda}$.

Theorem 2.1 *Let X be a real Banach space, $\eta : X \times X \to X$ be a Lipschitz continuous operator with constant $\tau > 0$, i.e., for all $x, y \in X$, $\|\eta(x,y)\| \leq \tau\|x-y\|$. Let $H : X \to X^*$ be a strongly η-monotone operator with constants $r > 0$, $M : X \to 2^{X^*}$ be a multivalued (H,η)-monotone operator. Then, the resolvent operator $R^{H,\eta}_{M,\lambda} : X^* \to X$ is Lipschitz continuous with constant $\frac{\tau}{r}$, that is,*

$$\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \leq \frac{\tau}{r}\|u-v\|, \quad \forall u, v \in X^*.$$

Proof We first show that $R^{H,\eta}_{M,\lambda}$ is single-valued operator. Let $\forall u \in X^*$, $x, y \in (H+\lambda M)^{-1}(u)$. It follows that $-Hx + u \in \lambda M(x)$ and $-Hy + u \in \lambda M(y)$. Since M is (H,η)-monotone, we have

$$0 \leq \langle(-Hx+u) - (-Hy+u), \eta(x-y)\rangle = \langle Hy - Hx, \eta(x-y)\rangle \leq -r\|x-y\|^2.$$

The strong η-monotonicity of H implies that $x = y$. Thus, $R^{H,\eta}_{M,\lambda}(u)$ is single-valued. The proof is completed. □

Let $u, v \in X^*$. It follows that $R^{H,\eta}_{M,\lambda}(u) = (H+\lambda M)^{-1}(u)$ and $R^{H,\eta}_{M,\lambda}(v) = (H+\lambda M)^{-1}(v)$. This implies that

$$\frac{1}{\lambda}(u - H(R^{H,\eta}_{M,\lambda}(u))) = M(R^{H,\eta}_{M,\lambda}(u)),$$
$$\frac{1}{\lambda}(v - H(R^{H,\eta}_{M,\lambda}(v))) = M(R^{H,\eta}_{M,\lambda}(v)).$$

Since M is η-monotone, we obtain

$$\begin{aligned}
&\frac{1}{\lambda}\langle(u - v - H(R^{H,\eta}_{M,\lambda}(u))) + H(R^{H,\eta}_{M,\lambda}(v)), \eta(R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v))\rangle \\
&= \frac{1}{\lambda}\langle(u - H(R^{H,\eta}_{M,\lambda}(u))) - (v - H(R^{H,\eta}_{M,\lambda}(v))), \eta(R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v))\rangle \\
&= \langle M(R^{H,\eta}_{M,\lambda}(u)) - M(R^{H,\eta}_{M,\lambda}(v)), \eta(R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v))\rangle \geq 0.
\end{aligned}$$

The inequality above implies that

$$\langle u-v, \eta(R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v))\rangle \geq \langle H(R_{M,\lambda}^{H,\eta}(u))-H(R_{M,\lambda}^{H,\eta}(v)), \eta(R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v))\rangle.$$

By strong η-monotonicity of H, one has

$$\begin{aligned}\tau\|u-v\|\|R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v)\| &\geq \|u-v\|\|\eta(R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v))\| \\ &\geq \langle u-v, \eta(R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v))\rangle \\ &\geq \langle H(R_{M,\lambda}^{H,\eta}(u))-H(R_{M,\lambda}^{H,\eta}(v)), \eta(R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v))\rangle \\ &\geq r\|R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v)\|^2,\end{aligned}$$

i.e.,

$$\|R_{M,\lambda}^{H,\eta}(u)-R_{M,\lambda}^{H,\eta}(v)\| \leq \frac{\tau}{r}\|u-v\|.$$

Based on Theorem 2.1, we discuss the following existence and iterative algorithm for solution of the system of variational inclusions

$$\begin{aligned}0 &\in F(a,b)+M(a),\\ 0 &\in G(a,b)+N(b),\end{aligned} \tag{2.1}$$

where $F : X\times X \to X^*$ and $G : X\times X\to X^*$ are single-valued operator, and $M : X \to 2^{X^*}$, $N : X \to 2^{X^*}$ is a multivalued one, which is the system of nonlinear variational inclusions considered by Fang and Huang[10]. It is easy to see that problem (2.1) includes many variational inequalities (inclusions) as special cases.

Some examples of problem (2.1) include the following.

(1) If $M(x) = \Delta_\eta\varphi(x)$ and $N(y) = \Delta_\eta\phi(y)$, where $\varphi : X \to R\cup\{+\infty\}$ and $\phi : X \to R\cup\{+\infty\}$ are two proper, η-subdifferentiable functionals, and $\Delta_\eta\varphi(x)$ and $\Delta_\eta\phi(y)$ denote the η-subdifferential operators of φ and ϕ (see [10]), respectively, then, problem (2.1) reduces to the following problem: find $(a,b)\in X\times X$, such that

$$\begin{aligned}&\langle F(a,b), \eta(x,a)\rangle+\varphi(x)-\varphi(a)\geq 0, \quad \forall x\in X,\\ &\langle G(a,b), \eta(y,b)\rangle+\phi(y)-\phi(b)\geq 0, \quad \forall y\in X,\end{aligned}$$

which is called a system of nonlinear variational-like inequalities.

(2) If $M(x) = \partial\varphi(x)$ and $N(y) = \partial\phi(y)$, for all $x, y \in X$, where $\varphi : X \to R\cup\{+\infty\}$ and $\phi : X \to R\cup\{+\infty\}$ are two proper, convex, lower semicontinuous functionals, and $\partial\varphi$ and $\partial\phi$ denote the η-subdifferential operators of φ and ϕ, respectively, then, problem (2.1) reduces to the following problem: find $(a,b)\in X\times X$, such that

$$\begin{aligned}&\langle F(a,b), x-a\rangle+\varphi(x)-\varphi(a)\geq 0, \quad \forall x\in X,\\ &\langle G(a,b), y-b\rangle+\phi(y)-\phi(b)\geq 0, \quad \forall y\in X,\end{aligned}$$

which is called a system of nonlinear variational inequalities.

(3) If $M(x) = \partial\delta_A(x)$ and $N(y) = \partial\delta_B(y)$, for all $x\in A$ and $y\in B$, where $A\subset X$ and $B\subset X$ are two nonempty, closed, and convex subsets, δ_A and δ_B denote the indicator functions

of A and B, respectively, i.e.,

$$\delta_A(x) = \begin{cases} 0 & x \in A \\ +\infty & x \notin A, \end{cases}$$

then, problem (2.1) reduces to the following system of variational inequalities: find $(a,b) \in A \times B$, such that

$$\langle F(a,b), x-a \rangle \geq 0, \quad \forall x \in A,$$

$$\langle G(a,b), y-b \rangle \geq 0, \quad \forall y \in B.$$

(4) If $F(x,y) = G(x,y) = T(x)$, where $T : X \to X^*$ is a single-valued operator, then, above case (3) reduces to the following standard nonlinear variational inequality problem: find an element $a \in A$, such that

$$\langle T(x), x-a \rangle \geq 0, \quad \forall x \in A.$$

Lemma 2.1 *Let X be a real Banach space, $\eta : X \times X \to X$ be a Lipschitz continuous operator with constant τ, $H_1, H_2 : X \to X^*$ be two strongly η-monotone operator with constants ν_1 and ν_2, respectively, and let $M : X \to 2^{X^*}$ be (H_1,η)-monotone and $N : X \to 2^{X^*}$ be (H_2,η)-monotone. Then, for any given $(a,b) \in X \times X$, (a,b) is a solution of problem (2.1) if and only if (a,b) satisfies*

$$\begin{aligned} a &= R_{M,\rho}^{H_1,\eta}[H_1(a) - \rho F(a,b)], \\ b &= R_{N,\lambda}^{H_2,\eta}[H_2(b) - \lambda G(a,b)], \end{aligned} \tag{2.2}$$

where λ and $\rho > 0$ are two constants.

Proof The conclusion directly follows from Definition 1.3 and Theorem 2.1.

Theorem 2.2 *Let X^* be q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a Lipschitz continuous operator with constant σ, $H_1 : X \to X^*$ be a strongly η-monotone, Lipschitz continuous operator with constants ν_1, τ_1 and $H_2 : X \to X^*$ be a strongly η-monotone, Lipschitz continuous operator with constants ν_2, τ_2. Let $M : X \to 2^{X^*}$ be (H_1,η)-monotone and $N : X \to 2^{X^*}$ be (H_2,η)-monotone. Let $F : X \times X \to X^*$ be an operator, such that, for any given $(a,b) \in X \times X$, $F(\cdot,b)$ is strongly accretive with respect to H_1 and Lipschitz continuous with constants r_1 and s_1, respectively, and $F(a,\cdot)$ is Lipschitz continuous with constant θ. Let $G : X \times X \to X^*$ be an operator, such that, for any given $(x,y) \in X \times X$, $G(x,\cdot)$ is strongly accretive with respect to H_2 and Lipschitz continuous with constant r_2 and s_2, respectively, and $G(\cdot,y)$ is Lipschitz continuous with constant ξ. Suppose there exist constants $\lambda, \rho > 0$, such that*

$$\begin{aligned} &\nu_2\sigma\sqrt[q]{\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q} + \nu_1\sigma\lambda\xi < \nu_1\nu_2, \\ &\nu_1\sigma\sqrt[q]{\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q} + \nu_2\sigma\rho\theta < \nu_1\nu_2. \end{aligned} \tag{2.3}$$

Then, problem (2.1) admits a unique solution.

Proof For any given λ and $\rho > 0$, define $T_\rho : X \times X \to X$ and $S_\lambda : X \times X \to X$ by

$$T_\rho(u,v) = R_{M,\rho}^{H_1,\eta}[H_1(u) - \rho F(u,v)],$$

$$S_\lambda(u,v) = R_{N,\lambda}^{H_2,\eta}[H_2(v) - \lambda G(u,v)], \tag{2.4}$$

for all $u, v \in X$.

For any $(u_1,v_1),(u_2,v_2) \in X \times X$, it follows from (2.4) and Theorem 2.1 that

$$\begin{aligned}\|T_\rho(u_1,v_1) - T_\rho(u_2,v_2)\| &= \|R_{M,\rho}^{H_1,\eta}[H_1(u_1) - \rho F(u_1,v_1)] - R_{M,\rho}^{H_1,\eta}[H_1(u_2) - \rho F(u_2,v_2)]\| \\ &\le \frac{\sigma}{\nu_1}\|H_1(u_1) - H_1(u_2) - \rho[F(u_1,v_1) - F(u_2,v_2)]\| \\ &\le \frac{\sigma}{\nu_1}\|H_1(u_1) - H_1(u_2) - \rho[F(u_1,v_1) - F(u_2,v_1)]\| + \\ &\quad \frac{\sigma\rho}{\nu_1}\|F(u_2,v_1) - F(u_2,v_2)\|, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\|S_\lambda(u_1,v_1) - S_\lambda(u_2,v_2)\| &= \|R_{N,\lambda}^{H_2,\eta}[H_2(v_1) - \lambda G(u_1,v_1)] - R_{N,\lambda}^{H_2,\eta}[H_2(v_2) - \lambda G(u_2,v_2)]\| \\ &\le \frac{\sigma}{\nu_2}\|H_2(v_1) - H_2(v_2) - \lambda[G(u_1,v_1) - G(u_2,v_2)]\| \\ &\le \frac{\sigma}{\nu_2}\|H_2(v_1) - H_2(v_2) - \lambda[G(u_1,v_1) - G(u_1,v_2)]\| + \\ &\quad \frac{\sigma\lambda}{\nu_2}\|G(u_1,v_2) - G(u_2,v_2)\|. \end{aligned} \tag{2.6}$$

By assumptions, we have

$$\begin{aligned}&\|H_1(u_1) - H_1(u_2) - \rho[F(u_1,v_1) - F(u_2,v_1)]\|^q \\ &\quad \le \|H_1(u_1) - H_1(u_2)\|^q + \rho^q c_q\|F(u_1,v_1) - F(u_2,v_1)\|^q - \\ &\quad\quad q\rho\langle F(u_1,v_1) - F(u_2,v_1), J_q^*(H_1(u_1) - H_1(u_2))\rangle \\ &\quad \le (\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q)\|u_1 - u_2\|^q, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\|H_2(v_1) - H_2(v_2) - \lambda[G(u_1,v_1) - G(u_1,v_2)]\|^q \\ &\quad \le \|H_2(v_1) - H_2(v_2)\|^q + \lambda^q c_q\|G(u_1,v_1) - G(u_1,v_2)\|^q - \\ &\quad\quad q\lambda\langle G(u_1,v_1) - G(u_1,v_2), J_q^*(H_1(v_1) - H_1(v_2))\rangle \\ &\quad \le (\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q)\|v_1 - v_2\|^q, \end{aligned} \tag{2.8}$$

where $J_q^* : X^* \to X^{**}$ is the generalized duality mapping on X^*.

Furthermore,

$$\|F(u_2,v_1) - F(u_2,v_2)\| \le \theta\|v_1 - v_2\|, \tag{2.9}$$

$$\|G(u_1,v_2) - G(u_2,v_2)\| \le \xi\|u_1 - u_2\|. \tag{2.10}$$

It follows from (2.5)–(2.10) that

$$\|T_\rho(u_1,v_1) - T_\rho(u_2,v_2)\| \le \frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q}\|u_1 - u_2\| + \frac{\sigma\rho\theta}{\nu_1}\|v_1 - v_2\|, \tag{2.11}$$

$$\|S_\lambda(u_1,v_1) - S_\lambda(u_2,v_2)\| \le \frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q}\|v_1 - v_2\| + \frac{\sigma\lambda\xi}{\nu_2}\|u_1 - u_2\|, \tag{2.12}$$

$$\begin{aligned}
&\|T_\rho(u_1,v_1)-T_\rho(u_2,v_2)\|+\|S_\lambda(u_1,v_1)-S_\lambda(u_2,v_2)\| \\
&\le \Big(\frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q-q\rho r_1+\rho^q c_q s_1^q}+\frac{\sigma\lambda\xi}{\nu_2}\Big)\|u_1-u_2\|+ \\
&\Big(\frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q-q\lambda r_2+\lambda^q c_q s_2^q}+\frac{\sigma\rho\theta}{\nu_1}\Big)\|v_1-v_2\| \\
&\le k(\|u_1-u_2\|+\|v_1-v_2\|), \qquad (2.13)
\end{aligned}$$

where

$$k=\max\Big\{\frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q-q\rho r_1+\rho^q c_q s_1^q}+\frac{\sigma\lambda\xi}{\nu_2},\frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q-q\lambda r_2+\lambda^q c_q s_2^q}+\frac{\sigma\rho\theta}{\nu_1}\Big\}.$$

Define $\|\cdot\|_1$ on $X\times X$ by

$$\|(u,v)\|_1=\|u\|+\|v\|,\quad \forall (u,v)\in X\times X.$$

It is easy to see that $(X\times X,\|\cdot\|_1)$ is a Banach space. For any given $\lambda>0$ and $\rho>0$, define $Q_{\lambda,\rho}: X\times X\to X\times X$ by

$$Q_{\lambda,\rho}(u,v)=\left(T_\rho(u,v),S_\lambda(u,v)\right),\quad \forall (u,v)\in X\times X.$$

By (2.3), we know that $0<k<1$. It follows from (2.13) that

$$\|Q_{\lambda,\rho}(u_1,v_1)-Q_{\lambda,\rho}(u_2,v_2)\|\le k\left(\|u_1-u_2\|+\|v_1-v_2\|\right).$$

This proves that $Q_{\lambda,\rho}: X\times X\to X\times X$ is a contraction operator. Hence, there exists a unique $(a,b)\in X\times X$, such that $Q_{\lambda,\rho}(a,b)=(a,b)$, namely,

$$\begin{aligned}
a&=R^{H_1,\eta}_{M,\rho}[H_1(a)-\rho F(a,b)],\\
b&=R^{H_2,\eta}_{N,\lambda}[H_2(b)-\lambda G(a,b)].
\end{aligned}$$

This completes the proof of Theorem 2.2. □

Motivated by above Theorem 2.2, we consider the following iterative algorithms.

Algorithm 2.1 Let η,H_1,H_2,M,N,F and G be as in Theorem 2.2. For any given $(a_0,b_0)\in X\times X$, define the iterative sequence $\{(a_n,b_n)\}$ by

$$\begin{aligned}
a_{n+1}&=R^{H_1,\eta}_{M,\rho}[H_1(a_n)-\rho F(a_n,b_n)],\\
b_{n+1}&=R^{H_2,\eta}_{N,\lambda}[H_2(b_n)-\lambda G(a_n,b_n)]. \qquad (2.14)
\end{aligned}$$

Theorem 2.3 *Let X^* be q-uniformly smooth Banach space, and $\eta,H_1,\ H_2,M,N,F$ and G be as in Theorem 2.2. Assume that all the conditions (2.3) of Theorem 2.2 are satisfied. Then $\{(a_n,b_n)\}$ generated by Algorithm 2.1 converges strongly to the unique solution (a,b) of problem (2.1) and there exists $0<d<1$, such that*

$$\|a_n-a\|+\|b_n-b\|\le d^n(\|a_0-a\|+\|b_0-b\|).$$

Proof By Theorem 2.2, problem (2.1) admits a unique solution, (a,b). It follows from Lemma 2.1 that

$$a=R^{H_1,\eta}_{M,\rho}[H_1(a)-\rho F(a,b)],$$

$$b = R_{N,\lambda}^{H_2,\eta}[H_2(b) - \lambda G(a,b)].$$

Thus

$$\begin{aligned}\|a_{n+1} - a\| &= \|R_{M,\rho}^{H_1,\eta}[H_1(a_n) - \rho F(a_n,b_n)] - R_{M,\rho}^{H_1,\eta}[H_1(a) - \rho F(a,b)]\| \\ &\leq \frac{\sigma}{\nu_1}\|H_1(a_n) - H_1(a)\| + \rho\|F(a_n,b_n) - F(a,b)\| \\ &\leq \frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q}\|a_n - a\| + \frac{\sigma\rho\theta}{\nu_1}\|b_n - b\|, \qquad (2.15)\end{aligned}$$

and

$$\begin{aligned}\|b_{n+1} - b\| &\leq \frac{\sigma}{\nu_2}\|H_2(b_n) - H_2(b)\| + \lambda\|G(a_n,b_n) - G(a,b)\| \\ &\leq \frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q}\|b_n - b\| + \frac{\sigma\lambda\xi}{\nu_2}\|a_n - a\|. \qquad (2.16)\end{aligned}$$

It follows from (2.15) and (2.16) that

$$\begin{aligned}\|a_{n+1} - a\| + \|b_{n+1} - b\| \leq &\Big(\frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q} + \frac{\sigma\lambda\xi}{\nu_2}\Big)\|a_n - a\| + \\ &\Big(\frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q} + \frac{\sigma\rho\theta}{\nu_1}\Big)\|b_n - b\| \\ \leq &k(\|a_n - a\| + \|b_n - b\|) \leq d^n(\|a_0 - a\| + \|b_0 - b\|), \qquad (2.17)\end{aligned}$$

where $0 < d = k < 1$ is defined by

$$k = d = \max\Big(\frac{\sigma}{\nu_1}\sqrt[q]{\tau_1^q - q\rho r_1 + \rho^q c_q s_1^q} + \frac{\sigma\lambda\xi}{\nu_2}, \frac{\sigma}{\nu_2}\sqrt[q]{\tau_2^q - q\lambda r_2 + \lambda^q c_q s_2^q} + \frac{\sigma\rho\theta}{\nu_1}\Big).$$

Thus, using formula (2.17) and iterating, we have $a_n \to a, b_n \to b$.

Remark 2.1 Applying above method, we can also construct the Mann iterative algorithm for $\{a_n\}$ and $\{b_n\}$ to approximate the unique solution of problem (2.1)

$$a_{n+1} = \alpha_n a_n + (1 - \alpha_n)R_{M,\rho}^{H_1,\eta}[H_1(a_n) - \rho F(a_n,b_n)], \quad n = 1,2,\ldots,$$

$$b_{n+1} = \alpha_n b_n + (1 - \alpha_n)R_{N,\lambda}^{H_2,\eta}[H_2(b_n) - \lambda G(a_n,b_n)], \quad n = 1,2,\ldots,$$

where $0 \leq \alpha_n < 1, \limsup \alpha_n < 1$. Similarly, we can prove the same results under the conditions of Theorem 2.2.

Remark 2.2 Conditions (2.3) hold for some suitable values of constants, because by [6] we have the following

Proposition *Let $E = l^P$ (or L^q), $1 < q \leq< \infty$, $x, y \in E$. We have*

(i) If $1 < q < 2$, then

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

where $c_q = (1 + b_q^{q-1})/(1 + b_q)^{q-1}$, and b_q is the unique solution of equation

$$(q - 2)b_q^{q-1} + (q - 1)(1 + b_q)^{q-1} - 1 = 0, \quad 0 < b_q < 1.$$

(ii) If $q \geq 2$, *then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + (q-1)\|y\|^2.$$

It is clear that when $1 < q < 2$, we have $0 < c_q < 1$; when $q = 2$, we have $c_q = 1$; when $q \geq 2$, we have $c_q \geq 1$, and the value of d is continuous function of q. Therefore, the value of d is attainable. For example, if one sets $q = 2.05$, $c_q = 1.05$, $\sigma = 0.8$, $\tau_1 = \tau_2 = 2$, $\rho = 0.5$, $r_1 = r_2 = 0.5$, $s_1 = s_2 = 0.5$, $\nu_1 = \nu_2 = 4$, $\lambda = 0.5$ and $\xi = \theta = 2$, then $d = 0.578$ and (2.3) is satisfied.

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