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A Note on Modified Mann Iterations for Zero Points of Accretive Operators

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Abstract A more general form of modified Mann iterations which converges strongly to a zero point of an m-accretive operator is given. The work in this paper is an extension and complement of the corresponding result of Kim T.H. and Xu H.K in 2005

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1. Introduction and preliminaries

It is known [1] that many physically significant problems can be modelled by initial-value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = x_0, \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrodinger equations. If in (1.1), $x(t)$ is independent of t, then equation (1.1) is reduced to

$$Au = 0,$$

whose solutions correspond to the equilibrium points (or zero points) of system (1.1). Consequently, considerable research efforts have been devoted, especially within the past 30 years or so, to iterative methods for approximating these equilibrium points [2−4].

Now, let X be a real Banach space and X^* be its dual space. The normalized duality mapping J from X into 2^{X^*} is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between X and X^*. It is well known that if X is smooth, then J is single-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

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Let C be a non-empty closed convex subset of X. Recall that a mapping $T : C \to C$ is called nonexpansive if the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T; that is,

$$\mathrm{Fix}(T) = \{x \in C : Tx = x\}.$$

Recall that a multi-valued operator A with domain $D(A)$ and range $R(A)$ in X is accretive if, for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2$, there exists a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j\rangle \geq 0.$$

An accretive operator A is m-accretive if $R(I + rA) = X$ for each $r > 0$. The set of zero points of A is denoted by $A^{-1}(0)$. That is,

$$A^{-1}(0) = \{z \in D(A) : 0 \in A(z)\}.$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$.

Lemma 1.1[5] *If A is m-accretive, then $J_r : X \to X$ is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}(0)$, for all $r > 0$.*

Lemma 1.2[6] *A Banach space X is uniformly smooth if and only if the duality mapping J is single-valued and norm-to-norm uniformly continuous on bounded subsets of X.*

Lemma 1.3[7] *In a Banach space X, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad x, y \in X, \tag{1.2}$$

where $j(x + y) \in J(x + y)$.

Recall that if C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is sunny[8] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$.

Lemma 1.4[3] *Let X be a uniformly smooth Banach space, let C be a nonempty closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly to a fixed point of T as $t \to 0$. Define $Q : C \to \mathrm{Fix}(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $\mathrm{Fix}(T)$; that is, Q satisfies the property:*

$$\langle u - Qu, j(z - Qu)\rangle \leq 0, \quad u \in C, \quad z \in \mathrm{Fix}(T).$$

Lemma 1.5[9] *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ *and* $\{\sigma_n\}_{n=0}^{\infty}$ *are sequences such that*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) either $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$. *Then* $\{a_n\}_{n=0}^{\infty}$ *converges to zero.*

In 2005, Kim and Xu obtained the following modified Mann iterative theorem in [4]:

Proposition 1.1[4] *Suppose* X *is a uniformly smooth Banach space and* A *is an* m*-accretive operator in* X *such that* $A^{-1}(0) \neq \emptyset$. *Given a point* $u \in X$ *and sequences* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ *and* $\{r_n\}_{n=0}^{\infty} \subset (0,+\infty)$. *Define an iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *in* X *by*

$$\begin{cases} x_0 = x \in X, \quad \text{arbitrarily}, \\ y_n = J_{r_n} x_n, \ n \ge 0, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \ n \ge 0. \end{cases} \tag{1.3}$$

Moreover, the following conditions are satisfied:

(i) $\alpha_n \to 0$ *as* $n \to \infty$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $r_n \ge \varepsilon$ *for some* $\varepsilon > 0$ *and for all* $n \ge 0$. *Also assume*

$$\sum_{n=0}^{\infty} |1 - \frac{r_n}{r_{n+1}}| < +\infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a zero point of* A.

In this paper, our purpose is to extend the iteration algorithm (1.3) studied in [4] to a more general case. And also show that as a special case, some conditions of Proposition 1.1 can be omitted.

2. Strong convergence to zero points of accretive operators

Theorem 2.1 *Suppose* X *is a uniformly smooth Banach space and* A *is an* m*-accretive operator in* X *such that* $A^{-1}(0) \neq \emptyset$. *Given a point* $u \in X$ *and sequences* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ *and* $\{r_n\}_{n=0}^{\infty} \subset (0,+\infty)$. *Define an iterative sequence* $\{x_n\}_{n=0}^{\infty}$ *in* X *by*

$$\begin{cases} x_0 \in X, \quad \text{arbitrarily}, \\ y_n = f(r_n, x_n, A), \ n \ge 0, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \ n \ge 0 \end{cases} \tag{2.1}$$

where $f : (0,+\infty) \times X \times \Gamma(X) \to X$ *and* $\Gamma(X) = \{A : X \to X | A \text{ is an } m\text{-accretive operator}\}$. *Moreover, suppose the following conditions are satisfied:*

(i) $\alpha_n \to 0$ *as* $n \to \infty$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) There exists a nonexpansive mapping $S : X \to X$ *such that* $\mathrm{Fix}(S) = A^{-1}(0)$ *and* $s - \lim_{n\to\infty}(y_n - Sy_n) = 0$;

(iii) $\|y_n - p\| \le \|x_n - p\|$, *for* $\forall p \in A^{-1}(0)$ *and* $n \ge 0$.

Then $\{x_n\}_{n=0}^{\infty}$ *strongly converges to a zero point of* A.

Proof First we will prove that $\{x_n\}_{n=0}^{\infty}$ is bounded.

Take $p \in A^{-1}(0)$, it follows from condition (iii) that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n \|u - p\| + (1 - \alpha_n)\|y_n - p\| \le \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\le \max\{\|u - p\|, \|x_n - p\|\}.\end{aligned}$$

Now, by induction, we get

$$\|x_{n+1} - p\| \le \max\{\|u - p\|, \|x_0 - p\|\}, \quad n \ge 0.$$

Hence $\{x_n\}$ is bounded, so is $\{y_n\}$. As a result of (i), we have

$$\|x_{n+1} - y_n\| = \alpha_n \|y_n - u\| \to 0, \quad \text{as} \quad n \to \infty.$$

So $s - \lim_{n\to\infty}(x_{n+1} - y_n) = 0$. Moreover, condition (ii) implies that $s - \lim_{n\to\infty}(x_{n+1} - Sx_{n+1}) = 0$.

Next, we claim that

$$\limsup_{n\to\infty} \langle u - q, j(x_n - q)\rangle \le 0, \tag{2.2}$$

where $\mathrm{Fix}(S) \ni q = s - \lim_{t\to 0} z_t$ with $z_t \in X$ being the fixed point of the contraction $X \ni z \to tu + (1 - t)Sz$ (see Lemma 1.4). Indeed, z_t solves the fixed point equation

$$z_t = tu + (1 - t)Sz_t.$$

Thus we have

$$z_t - x_n = (1 - t)(Sz_t - x_n) + t(u - x_n).$$

We apply Lemma 1.3 to get

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1 - t)^2\|Sz_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n)\rangle \\ &\le (1 - 2t + t^2)\|z_t - x_n\|^2 + a(t) + 2t\langle u - z_t, j(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,\end{aligned}$$

where

$$a(t) = (2\|z_t - x_n\| + \|x_n - Sx_n\|)\|x_n - Sx_n\| \to 0, \quad \text{as} \quad n \to \infty. \tag{2.3}$$

It follows that

$$\langle z_t - u, j(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}a(t). \tag{2.4}$$

Let $n \to \infty$ in (2.4) and notice (2.3), we have

$$\limsup_{n\to\infty} \langle z_t - u, j(z_t - x_n)\rangle \le \frac{t}{2}M, \tag{2.5}$$

where $M > 0$ is a constant such that $M \ge \|z_t - x_n\|^2$ for all $t \in (0, 1)$ and $n \ge 0$. Since $\{z_t - x_n\}$ is bounded, the duality map j is norm-to-norm uniformly continuous on bounded sets of X and z_t strongly converges to q. It follows (by letting $t \to 0$ in (2.5)) that (2.2) is available.

Finally, we show that $s - \lim_{n\to\infty} x_n = q$.

Indeed, since $\mathrm{Fix}(S) = A^{-1}(0)$, using Lemma 1.3 and the assumption that $\|y_n - p\| \le \|x_n - p\|$, for $\forall p \in A^{-1}(0)$, we obtain

$$\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(y_n - q) + \alpha_n(u - q)\|^2$$

$$\leq (1-\alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\langle u-q, j(x_{n+1}-q)\rangle$$
$$\leq (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u-q, j(x_{n+1}-q)\rangle.$$

Now, we apply Lemma 1.5 and use (2.2) to see that $\|x_n - q\| \to 0$, as $n \to \infty$. This completes the proof. $\square$

Remark 2.1 If we take $S \equiv J_r$ and $y_n = J_{r_n}x_n$ in ALG(2.1) of Theorem 2.1, where $n \geq 0$ and $\{r_n\}_{n=0}^{\infty} \subset (0,1)$, then we can know that Proposition 1.1 is a special case of Theorem 2.1. Moreover, in this case, the condition that $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ in Proposition 1.1 is not necessary.

Remark 2.2 In view of the definitions of strongly accretive operator with constant r and H-accretive operator introduced in [10], we can get another example. That is, if $H : X \to X$ is nonexpansive and strongly accretive with constant $r \equiv 1$, and A is H-accretive, then take $S = (H+\lambda A)^{-1}H$ and $y_n = (H+\lambda A)^{-1}Hx_n$ in ALG(2.1) of Theorem 2.1, where $\lambda > 0$. Theorem 2.3 in [10] guarantees that the result of Theorem 2.1 is still true.

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