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Existence of Solutions of a Nonlinear Three-Point Boundary Value Problem for Third-Order Ordinary Differential Equations

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Abstract In this paper, existence of solutions of third-order differential equation

$$y'''(t) = f(t, y(t), y'(t), y''(t))$$

with nonlinear three-point boundary condition

$$\begin{cases} g(y(a), y'(a), y''(a)) = 0, \\ h(y(b), y'(b)) = 0, \\ I(y(c), y'(c), y''(c)) = 0 \end{cases}$$

is obtained by embedding Leray-Schauder degree theory in upper and lower solutions method, where $a, b, c \in R, a < b < c$; $f : [a, c] \times R^3 \to R, g : R^3 \to R, h : R^2 \to R$ and $I : R^3 \to R$ are continuous functions. The existence result is obtained by defining the suitable upper and lower solutions and introducing an appropriate auxiliary boundary value problem. As an application, an example with an explicit solution is given to demonstrate the validity of the results in this paper.

Keywords Existence of solutions; three-point boundary value problems; upper and lower solutions method; Leray-Schauder degree theory.

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1. Introduction

Boundary value problems for third-order ordinary differential equations have received much attention in the last few decades because of the theoretical challenges involved in the investigation of such problems and also because of its importance in practical applications such as boundary layer theory in fluid mechanics[1]. Many methods, such as upper and lower solutions method[1−7], shooting method[8], monotone iterative method[9] had been developed to derive existence of solutions for third-order differential equations with various boundary conditions

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including two-point boundary condition[1−8], three-point boundary condition[10] and periodic boundary condition[9], etc. Recently, considerable attention was paid to third-order boundary value problems with nonlinear boundary conditions, but most of works concentrate on two-point boundary value problems. Multi-point boundary value problems are still largely unexplored.

In this paper, existence of solutions of a nonlinear three-point boundary value problem for third-order differential equations is studied using upper and lower solutions method along with Leray-Schauder degree theory. Upper and lower solutions method was developed by Nagumo[11] for deducing existence of solutions of second-order Dirichlet boundary value problems. Ever since its derivation, this method was widely adopted to obtain existence of solutions for various boundary value problems for second-order[12], third-order[1−7] and higher-order differential equations[13−15]. In recent years, upper and lower solutions method was further developed at two aspects. On the one hand, this method is combined with other skills, such as maximum principle[3], Leray-Schauder degree theory[4−7], monotone iterative method[9,14], which can be used to derive existence of solutions of considered problems more simply and directly. On the other hand, the sign-Nagumo condition was proposed to loosen the restriction of expressions of differential equations[6−7]. In this work, we combine upper and lower solutions method with Leray-Schauder degree theory to deduce existence of solutions for third-order differential equations

$$y'''(t) = f(t, y(t), y'(t), y''(t)) \tag{1}$$

with nonlinear three-point boundary condition

$$\begin{cases} g(y(a), y'(a), y''(a)) = 0, \\ h(y(b), y'(b)) = 0, \\ I(y(c), y'(c), y''(c)) = 0. \end{cases} \tag{2}$$

The article proceeds as follows. In Section 2, the definitions of upper and lower solutions for boundary value problem (BVP)(1)–(2) and the Nagumo condition are given. Several assumptions which are needed in the proof are also presented. Section 3 is devoted to the main results of the paper. An example is given in the final section to demonstrate the validity of the results in the paper.

2. Preliminaries

Definition 1 *Functions $\alpha(t)$ and $\beta(t) \in C^3[a,b]$ are called the lower and upper solutions of BVP (1)–(2), respectively, if*

$$\alpha'''(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t)), \tag{3}$$

$$\beta'''(t) \leq f(t, \beta(t), \beta'(t), \beta''(t)) \tag{4}$$

and

$$\alpha'(t) \leq \beta'(t), t \in [a,c]; \alpha(t) \leq \beta(t), t \in [b,c]; \beta(t) \leq \alpha(t), t \in [a,b],$$

$$g(\beta(a), \beta'(a), \beta''(a)) \leq 0 \leq g(\alpha(a), \alpha'(a), \alpha''(a)), \tag{5}$$

$$h(\alpha(b), \alpha'(b)) \le 0 \le h(\beta(b), \beta' b)), \tag{6}$$

$$I(\alpha(c), \alpha'(c), \alpha''(c)) \le 0 \le I(\beta(c), \beta'(c), \beta''(c)). \tag{7}$$

Definition 2 *Third-order differential Eq. (1) is said to satisfy the Nagumo condition in a bounded region $D \subset [a, c] \times R^3$ if the right-hand side function of Eq. (1) is continuous and there exists a function $\varphi \in C[[0, \infty), (m, \infty)]$ such that*

$$|f(t, y(t), y'(t), y''(t))| \le \varphi(|y''(t)|), \tag{8}$$

where m is a positive constant and for any $\mu > 0$, $\int_\mu^{+\infty} \frac{s}{\varphi(s)} \mathrm{d}s = +\infty$.

To derive the results of this paper, the following assumptions are needed.

H_1: There exist the upper and lower solutions of BVP (1)-(2) defined like Definition 1;

H_2: Eq.(1) satisfies the Nagumo condition, and is increasing in y in the region $D_1 = [b, c] \times [\beta(t), \alpha(t)] \times R^2$ and decreasing in y in $D_2 = [a, b] \times [\alpha(t), \beta(t)] \times R^2$;

H_3: $g(y, y', y'')$ is continuous with respect to all its arguments in R^3, decreasing in y and increasing in y'';

H_4: $I(y, y', y'')$ is continuous with respect to all its arguments in R^3, decreasing in y and increasing in y'';

H_5: $h(y, y')$ is continuous with respect to y and y' and increasing in y'.

3. Main results

To attain the existence of solutions of BVP (1)–(2), we first introduce a boundary value problem consisting of Eq. (1) and the following nonlinear three-point boundary condition

$$\begin{cases} g(y(a), y'(a), y''(a)) = 0, \\ y(b) = 0, \\ I(y(c), y'(c), y''(c)) = 0. \end{cases} \tag{9}$$

BVP (1) and (9) is called the auxiliary boundary value problem.

Consider boundary value problem below with homotopy character

$$y''' = \lambda f(t, r(y), s(y'), y'') + (1 - \lambda) y' + \lambda (y' - s(y')) \varphi(|y''|), \tag{10}$$

$$y'(a) = \lambda \left[g(r_2(y(a)), s(y'(a)), y''(a)) + s(y'(a)) \right], \tag{11}$$

$$y(b) = 0, \tag{12}$$

$$y'(c) = \lambda \left[-I(r_1(y(c)), s(y'(c)), y''(c)) + s(y'(c)) \right], \tag{13}$$

where $\lambda \in [0, 1]$ and

$$r_1(y(t)) = \begin{cases} \alpha(t), & y < \alpha(t), \\ y(t), & \alpha(t) \le y \le \beta(t), \quad t \in [b, c]; \\ \beta(t), & y > \beta(t) \end{cases}$$

$$r_2(y(t)) = \begin{cases} \beta(t), & y < \beta(t), \\ y(t), & \beta(t) \le y \le \alpha(t), \quad t \in [a,b]; \\ \beta(t), & y > \alpha(t) \end{cases}$$

$$r(y(t)) = \begin{cases} r_1(y(t)), & t \in [a,b], \\ r_2(y(t)), & t \in [b,c], \end{cases}$$

$$s(y'(t)) = \begin{cases} \alpha'(t), & y < \alpha'(t), \\ y'(t), & \alpha'(t) \le y \le \beta('t), \quad t \in [a,c]. \\ \beta'(t), & y > \beta'(t) \end{cases}$$

In what follows, the priori estimates of solutions of BVP (10)–(13) are deduced first. In so doing, existence of solutions of BVP (10)–(13) with $\lambda = 1$ can be obtained by applying Leray-Schauder degree theory. It follows that existence of solutions of BVP (1) and (9) can be attained by performing the estimates of solutions and its derivatives of BVP (10)–(13) with $\lambda = 1$. Consequently, existence of solutions of BVP (1)–(2) can be obtained.

Lemma 1 *Suppose that H_1–H_4 hold. Then every solution of BVP (10)–(13) satisfies*

$$|y(t)| < \max\{c-b, b-a\} \times K, \quad |y'(t)| < K, \quad |y''(t)| < K_1, \tag{14}$$

where K and K_1 are both positive constants.

Proof According to the continuity of the functions f, g, h and I, and the boundedness of $\alpha(t)$ and $\beta(t)$ and their derivatives in bounded interval, we can choose a sufficiently large constant $K > 0$, such that for $t \in [a,c]$, the following inequalities hold

$$-K < \alpha'(t) \le \beta'(t) < K, \tag{15}$$

$$-f(t, \alpha(t), \alpha'(t), 0) - [K + \alpha'(t)]\varphi(0) < 0, \tag{16}$$

$$f(t, \beta(t), \beta'(t), 0) + [K - \beta'(t)]\varphi(0) > 0, \tag{17}$$

$$g(\beta(a), \beta'(a), 0) + \beta'(a) < K, \tag{18}$$

$$-I(\beta(c), \beta'(c), 0) + \beta'(c) < K, \tag{19}$$

$$-K < g(\alpha(a), \alpha'(a), 0) + \alpha'(a), \tag{20}$$

$$-K < -I(\alpha(c), \alpha'(c), 0) + \alpha'(c). \tag{21}$$

We first turn to prove $|y'(t)| < K$. Suppose that $|y'(t)| < K$ does not hold uniformly for $t \in [a,c]$. Then there exists at least a $t \in [a,c]$, such that either $y'(t) > K$ or $y'(t) < -K$ holds.

If the former case emerges, we define $\max_{t \in [a,c]} y'(t) = y'(t_0) \ge K > 0$. If $t_0 \in (a,c)$, it follows that $y'(t_0) \ge K > \beta'(t_0), y''(t_0) = 0, y'''(t_0) \le 0$. On the other hand, according to H_2, the definitions of $r(y)$ and $s(y')$ and inequality (17), for $\lambda \in (0,1]$, we have

$$\begin{aligned} y'''(t_0) &= \lambda f(t_0, r(y(t_0)), s(y'(t_0)), 0) + (1-\lambda)y'(t_0) + \lambda[y'(t_0) - s(y'(t_0))]\varphi(0) \\ &\ge \lambda f(t_0, \beta(t_0), \beta'(t_0), 0) + (1-\lambda)y'(t_0) + \lambda[K - \beta'(t_0)]\varphi(0) \end{aligned}$$

$$\geq \lambda[f(t_0, \beta(t_0), \beta'(t_0), 0) + [K - \beta'(t_0)]\varphi(0)] > 0,$$

which contradicts $y'''(t_0) \leq 0$. For $\lambda = 0$, it follows from Eq. (10) that

$$0 \geq y'''(t_0) = y'(t_0) > 0.$$

It is a contradiction too. Thus, $t_0 \notin (a, c)$ for $\lambda \in [0, 1]$.

If $t_0 = a$, then $\max_{t\in[a,c]} y'(t) = y'(a) \geq K > \beta'(a)$, which yields $y''(a^+) = y''(a) \leq 0$. For $\lambda \in (0, 1]$, it follows from equality (11), inequality (18) and H_3 that

$$\begin{aligned} K \leq y'(a) &= \lambda[g(r_2(y(a)), s(y'(a)), y''(a)) + s(y'(a))] \\ &\leq \lambda[g(\beta(a), \beta'(a), 0) + \beta'(a)] < \lambda K < K. \end{aligned}$$

It is a contradiction. For $\lambda = 0$, it follows from equality (11) that $0 = y'(a) \geq K > 0$. It is a contradiction too. Thus, $t \neq a$.

Similarly, if $t_0 = c$, then $\max\limits_{t\in[a,c]} y'(t) = y'(c) \geq K > \beta'(c)$, which also yields $y''(c^-) = y''(c) \geq 0$. Hence, for $\lambda \in (0, 1]$, it can be deduced from equality (13), inequality (19) and H_5 that

$$\begin{aligned} K \leq y'(c) &= \lambda[-I(r_1(y(c)), s(y'(c)), y''(c)) + s(y'(c))] \\ &\leq \lambda[-I(\beta(c), \beta'(c), 0) + \beta'(c)] < \lambda K < K. \end{aligned}$$

For $\lambda = 0$, it follows from equality (13) that $0 = y'(c) \geq K > 0$. Thus, $t \neq c$.

Accordingly, $y'(t) > K$ is not possible for any $t \in [a, c]$. Similarly, we can deduce that $y'(t) < -K$ is impossible too for any $t \in [a, c]$. Now, $|y'(t)| < K$ is proved. Recalling $y(b) = 0$ and then integrating $y'(t)$ from b to t yields $|y(t)| < \max\{c - b, b - a\} \times K$.

As far as $|y''(t)| < K_1$ is concerned, note that

$$\begin{aligned} &|\lambda f(t, r(y), s(y'), y'') + (1 - \lambda)y' + \lambda(y' - s(y'))\varphi(|y''|)| \\ &\quad \leq \varphi(|y''|) + |y'| + (|y'| + |s(y')|)\varphi(|y''|) \leq K + (1 + 2K)\varphi(|y''|) \end{aligned}$$

and for any $\mu > 0$,

$$\int_\mu^{+\infty} \frac{s\mathrm{d}s}{K + (1 + 2K)\varphi(s)} < \int_\mu^{+\infty} \frac{s\mathrm{d}s}{(1 + 2K)\varphi(s)} = \frac{1}{(1 + 2K)} \int_\mu^{+\infty} \frac{s\mathrm{d}s}{\varphi(s)} = +\infty.$$

That is, the right-hand side function of Eq. (10) satisfies the Nagumo condition. Thus, $|y''(t)| < K_1$ can be attained according to the Lemma 1 in [5].

Lemma 2 *Suppose that H_1–H_4 hold. Then BVP (10)–(13) with $\lambda = 1$ has at least one solution satisfying*

$$\alpha'(t) \leq y'(t) \leq \beta'(t), t \in [a, c]; \alpha(t) \leq y(t) \leq \beta(t), t \in [b, c]; \beta(t) \leq y(t) \leq \alpha(t), t \in [a, b].$$

Proof Define two operators as follows

$$L = (y''', y(a'), y'(c)) : \quad D(L) \to C([a, c], R),$$

where $D(L) = \{y(t) \in C^3[a, c] | y(b) = 0\}$, and

$$T_\lambda = (N_\lambda, A_\lambda, B_\lambda) : C^2([a, c], R) \to C([a, c], R),$$

where

$$N_\lambda = \lambda f(t, r(y), s(y'), y'') + (1-\lambda)y' + \lambda(y' - s(y'))\varphi(|y''|),$$

$$A_\lambda = \lambda[g(r_2(y(a)), s(y'(a)), y''(a)) + s(y'(a))],$$

$$B_\lambda = \lambda[-I(r_1(y(c)), s(y'(c)), y''(c)) + s(y'(c))].$$

Therefore, BVP (10)–(13) is equivalent to the following operator equation

$$[I - L^{-1}T_\lambda]y = 0,$$

where I denotes the unit operator. Obviously, $L^{-1}T_\lambda$ is a completely continuous operator.

Define a bounded open domain

$$\Omega = \{y(t) \in C^2[a,c] \,|\, |y(t)| < \max\{c-b, b-a\} \times K, \quad |y'(t)| < K, \quad |y''(t)| < K_1\}.$$

Then, it follows from Lemma 1 that $[I - L^{-1}T_\lambda]\partial\Omega \neq 0$. Hence, the degree $\mathrm{Deg}(I - L^{-1}T_\lambda, \Omega, 0)$ is well defined. The invariance of degree under homotopy yields

$$\mathrm{Deg}(I - L^{-1}T_0, \Omega, 0) = \mathrm{Deg}(I - L^{-1}T_1, \Omega, 0).$$

Since the operator equation $[I - L^{-1}T_0]y = 0$ is equivalent to boundary value problem

$$y''' = y', \quad y'(a) = 0, \quad y(b) = 0, \quad y'(c) = 0, \tag{22}$$

which has only a trivial solution, existence of solutions of BVP (10)–(13) with $\lambda = 1$ is obvious.

Next, we turn to prove the later part of Lemma 2. First, we deduce $y'(t) \leq \beta'(t)$. Suppose that $y'(t) \leq \beta'(t)$ does not hold uniformly for $t \in [a,c]$. Then there exists at least a $t \in [a,c]$ such that $y'(t) > \beta'(t)$. Define $\max_{t\in[a,c]}[y'(t) - \beta'(t)] = y'(t_1) - \beta'(t_1) > 0$.

If $t_1 \in (a,c)$, then $y'(t_1) > \beta'(t_1), y''(t_1) = \beta''(t_1)$ and $y'''(t_1) \leq \beta'''(t_1)$. On the other hand,

$$\begin{aligned} y'''(t_1) - \beta'''(t_1) \geq & f(t_1, r(y(t_1)), s(y'(t_1)), y''(t_1)) + \\ & [y'(t_1) - s(y'(t_1))]\varphi(|y''(t_1)|) - f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) \\ > & f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) - f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) = 0, \end{aligned}$$

which contradicts $y'''(t_1) \leq \beta'''(t_1)$. Thus, $t_1 \notin (a,c)$.

If $t_1 = a$, then $\max_{t\in[a,c]}[y'(t) - \beta'(t)] = y'(a) - \beta'(a) > 0$, which results in $y''(a^+) - \beta''(a^+) = y''(a) - \beta''(a) \leq 0$. It then follows from inequality (5), equality (11) and H_3 that

$$\begin{aligned} y'(a) &= g[r_2(y(a)), s(y'(a)), y''(a)] + s(y'(a)) \\ &\leq g[\beta(a), \beta'(a), \beta''(a)] + \beta'(a) \leq \beta'(a). \end{aligned}$$

It is a contradiction. Thus, $t_1 \neq a$.

By similar deduction, $t_1 \neq c$ can be concluded.

Now, we can see that $y'(t) \leq \beta'(t)$ holds uniformly for $t \in [a,c]$. By the similar way, we can prove that $y'(t) \geq \alpha'(t)$ also holds uniformly for $t \in [a,c]$. The integration of $\alpha'(t) \leq y'(t) \leq \beta'(t)$ from b to t if $t \in [b,c]$ and from t to b if $t \in [a,b]$ yields $\alpha(t) \leq y(t) \leq \beta(t), t \in [b,c]$ and $\beta(t) \leq y(t) \leq \alpha(t), t \in [a,b]$ respectively. The proof of Lemma 2 is completed. □

Theorem 1 *Assume that H_1–H_4 hold. Then BVP (1) and (9) has at least one solution $y(t) \in C^3[a,c]$ satisfying*

$$\alpha'(t) \le y'(t) \le \beta'(t), t \in [a,c]; \alpha(t) \le y(t) \le \beta(t), t \in [b,c]; \beta(t) \le y(t) \le \alpha(t), t \in [a,b].$$

Proof From the latter part of Lemma 2, we know that BVP (10)–(13) with $\lambda = 1$ is equivalent to BVP (1) and (9). Thus, the conclusion of Theorem 1 is obvious.

Theorem 2 *Suppose that H_1–H_5 hold. Then BVP (1)–(2) has at least one solution $y(t) \in C^3[a,c]$ satisfying*

$$\alpha'(t) \le y'(t) \le \beta'(t), t \in [a,c]; \alpha(t) \le y(t) \le \beta(t), t \in [b,c]; \beta(t) \le y(t) \le \alpha(t), t \in [a,b].$$

Proof Assume that $y_1(t) \in C^3[a,c]$ is a solution of BVP (1) and (9). Then, it follows from Lemma 2 that $\alpha(b) = y_1(b) = \beta(b)$. By simple argument, $\beta'(b) \ge y_1(b) \ge \alpha'(b)$ can be attained. Hence, by H_5, we have

$$h(\beta(b), \beta'(b)) \ge h(y_1(b), y_1'(b)) \ge h(\alpha(b), \alpha'(b)).$$

Recalling inequality (6) yields $h(y_1(b), y_1'(b)) = 0$. Therefore, $y_1(t) \in C^3[a,c]$ is also a solution of BVP (1)–(2). Thus, the result of Theorem 2 is obtained. □

3. Example

Consider a nonlinear third-order ordinary differential equation

$$y''' = -ty + t^2 y' + (y'')^2 \sin(y')^2 := f(t, y, y', y'') \tag{23}$$

with the following nonlinear three-point boundary condition

$$\begin{cases} -y(-1) + a(y'(-1))^3 + b(y''(-1))^{2n+1} = 0, \\ y(0) + (y'(0))^2 = 0, \\ -y(1) + c(y'(1))^3 + d(y''(1))^{2n+1} = 0, \end{cases} \tag{24}$$

where $n = 1, 2, \ldots, a, b, c, d \in R$.

The upper and lower solutions of BVP (23)–(24) can be taken as $\alpha(t) = -t$ and $\beta(t) = t$, respectively. Consequently,

$$-1 = \alpha'(t) \le \beta'(t) = 1, t \in [-1,1]; \alpha(t) \le \beta(t), t \in [0,1]; \beta(t) \le \alpha(t), t \in [-1,0]$$

hold. Furthermore, by simple arguments, it can be verified that inequalities (3)–(7) and H_3–H_5 hold if $a \le -1, b > 0$ and $c \ge 1, d > 0$. Since $f_y(t, y, y', y'') = -t$, H_2 holds if $D_1 = [-1,0] \times [t,-t] \times R^2$ and $D_2 = [0,1] \times [-t,t] \times R^2$ are defined. Now, we can see that H_1–H_5 all hold. Hence, according to Theorem 2, BVP (23)–(24) has at least one solution $y(t) \in C^3[a,c]$ satisfying

$$-1 \le y'(t) \le 1, t \in [-1,1]; t \le y(t) \le -t, t \in [-1,0]; -t \le y(t) \le t, t \in [0,1]. \tag{25}$$

Note that $y(t) \equiv 0$ is a solution of BVP (23)–(24), which obviously satisfies inequalities (25). In this manner, the validity of the results in the paper can be seen.

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