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A High-Order Dikin-Type Algorithm for $P_*(\kappa)$-LCPs in a Wide Neighborhood of the Central Path

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Abstract Based on the idea of Dikin-type primal-dual affine scaling method for linear programming, we describe a high-order Dikin-type algorithm for $P_*(\kappa)$-matrix linear complementarity problem in a wide neighborhood of the central path, and its polynomial-time complexity bound is given. Finally, two numerical experiments are provided to show the effectiveness of the proposed algorithms.

Keywords complementarity problem; high-order affine scaling; polynomial-time complexity; interior-point algorithm; $P_*(\kappa)$-matrix.

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1. Introduction

In this paper, we consider the LCP as follows: to find a pair $(x,s)\in R^n\times R^n (n\geq 2)$, such that

$$s=Mx+q,(x,s)\geq 0 \ \text{ and } \ x^Ts=0, \tag{1}$$

where $M\in R^{n\times n}$ is a $P_*(\kappa)$-matrix and $q\in R^n$. A matrix $M\in R^{n\times n}$ is said to be a $P_*(\kappa)$-matrix if there exists a constant $\kappa\geq 0$ such that

$$x^TMx\geq -4\kappa\sum_{x\in I}x_i[Mx]_i,$$

where $x\in R^n$, $I=\{i:1\leq i\leq n, x_i[Mx]_i\geq 0\}$.

Note that the class of $P_*(\kappa)$-matrices contains the class PSD of positive semi-definite matrices, i.e., matrices M satisfying $x^TMx\geq 0$ for all $x\in R^n$, and the class P-matrices with all the principal minors positive.

Let $\Omega=\{(x,s)|s=Mx+q,(x,s)\geq 0\}$ and $\Omega^+=\{(x,s)|(x,s)\in\Omega,(x,s)>0\}$. Then they are called feasible solutions and interior feasible solutions of (1), respectively.

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Since Karmarkar published the first paper on interior point method[1] in 1984, interior point method has received a great deal of attention due to its polynomial complexity and excellent practicality and it has been introduced to linear programming and monotone linear complementarity problem[2]. These algorithms can be classified into three main groups: projective algorithms[3,4]; affine scaling algorithms, originally proposed by Dikin[5]; path following algorithms[6,7].

Path-following algorithms generate a sequence of points within certain neighborhoods of the central path C, which is defined as follows:

$$C = \{(x, s) \in \Omega^+ | Xs = \mu e, \quad \text{where} \quad \mu = x^T s/n\}.$$

The function of C is to prevent iterates from prematurely getting too close to the boundary of the feasible region. Depending on the norm used in their definition, these neighborhoods include:

$$N_2(\beta) = \{(x, s) \in \Omega^+ | \parallel Xs - \mu e \parallel \leq \beta\mu, \text{ where } \mu = x^T s/n\},$$
$$N_\infty(\beta) = \{(x, s) \in \Omega^+ | \parallel Xs - \mu e \parallel_\infty \leq \beta\mu, \text{ where } \mu = x^T s/n\},$$

where β is any fixed constant in $(0, 1)$.

Algorithms based on the larger $N_\infty(\beta)$ neighborhood are called long-step algorithms, and those based on the smaller $N_2(\beta)$ are called short-step algorithms. The theoretical iteration complexity of the short-step algorithms is $O(\sqrt{n}L)$ and the complexity of the long-step algorithms is at least $O(nL)$. In contrast, long-step algorithms outperform short-step ones by a big margin in practice. It seems that smaller neighborhoods generally restrict all iterates moved by a short step and they might be too conservative for solving real problems.

A second important class of algorithms are the affine scaling ones. The idea of the algorithms was first introduced by Dikin[5]. One of the main distinction between affine scaling and path-following algorithms is that the former do not use explicit centering directions. Moreover, the search-direction in affine scaling algorithms only depends on the current iterate, while the path-following methods use "target-points" on a path in the feasible region to compute the direction. The affine scaling algorithms are conceptually simpler.

In 1994, Huang and Ye[8] developed an r-order predictor-corrector primal-dual algorithm. The algorithm is based on an even larger neighborhood, namely,

$$N_\infty^-(\beta) = \{(x, s) \in \Omega^+ | Xs \geq (1 - \beta)\mu e, \quad \text{where} \quad \mu = x^T s/n\}.$$

It was shown that its iteration complexity is $O(n^{\frac{r+1}{2r}}L)$, where $r \in [1, n]$. Note that if $r = n$, then this iteration bound is asymptotically $O(\sqrt{n}L)$ as n increases.

Obviously, for any $\beta \in (0, 1)$, the following relation holds:

$$C \subseteq N_2(\beta) \subseteq N_\infty(\beta) \subseteq N_\infty^-(\beta) \subseteq \Omega^+.$$

Based on the idea of primal-dual Dikin affine scaling algorithm for linear programming and wide neighborhood interior-point algorithm for LCP[9], we describe a high-order Dikin-type algorithm for $P_*(\kappa)$-matrix linear complementarity problem in a wide neighborhood of the central. Finally, two numerical experiments are provided to show the effectiveness of the proposed algorithms.

The following notions will be used throughout this paper, the subscript T denotes transpose. Small letters denote scalar variables or vectors. While large letters denote matrices. The symbol $\|\cdot\|_1$ stands for 1-norm.

2. The r-order algorithm

To find an approximate solution for (1), we choose $(x,s)\in N_\infty^-(\beta)$, and solve the following linear system:

$$\Delta s^{(k)} = M\Delta x^{(k)}, \tag{2}$$

$$S\Delta x^{(k)}+X\Delta s^{(k)} = \begin{cases} \frac{-W^2e}{\|w\|}, & k=1, \quad (3)\\ -\sum_{t=1}^{k-1}\Delta X^t\Delta s^{k-t}, & k\geq 2 \quad (3')\end{cases}$$

to get the search direction $w=(\Delta x^k,\Delta s^k)$, and $w=Xs$. Let

$$x(\alpha)=x^k+\sum_{k=1}^{r}\alpha^k\Delta x^k,\ s(\alpha)=s^k+\sum_{k=1}^{r}\alpha^k\Delta s^k,\ \mu(\alpha)=\frac{x(\alpha)^{\mathrm{T}}s(\alpha)}{n}$$

denote the approximate solution of (1).

Now we are ready to state our algorithm:

Step 1. We take an arbitrary initial point $(x^0,s^0)\in N_\infty^-(\beta),\beta\in(0,1)$, let $k=1$;

Step 2. If $(x^k)^{\mathrm{T}}s^k\leq\varepsilon$ holds, then stops;

Step 3. Choose $(x,s)=(x^k,s^k)$, to get the search direction $\Delta w=(\Delta x^{(k)},\Delta s^{(k)})$;

Step 4. Let $\alpha=n^{-\frac{1}{2r}}\cdot\frac{1-\beta}{16n}\cdot\sqrt[r]{4\beta}(2\kappa+1)^{-2}$. Calculate $x^{k+1}=x^k+\sum_{k=1}^{r}\alpha^k\Delta x^{(k)}, s^{k+1}=s^k+\sum_{k=1}^{r}\alpha^k\Delta s^{(k)}$. Let $k:=k+1$, and go to Step 2.

3. Global convergence

For the convenience, we denote $(x,s)=(x^k,s^k)$. Without loss of generality, we may assume that $\mu=1$ in the following Lemmas.

Lemma 1 *Suppose $(x,s)\in\Omega^+,\Delta w=(\Delta x^{(j)},\Delta s^{(j)})\,(j=1,2,\ldots,r)$ is the solution of systems (2), (3), (3′) and let $\alpha>0$, $x(\alpha)=x+\sum_{j=1}^{r}\alpha^j\Delta x^{(j)}$, $s(\alpha)=s+\sum_{j=1}^{r}\alpha^j\Delta s^{(j)}$. We have*

$$X(\alpha)s(\alpha)=w-\frac{\alpha W^2e}{\|w\|}+\sum_{j=r+1}^{2r}(\alpha^j\sum_{t=j-r}^{r}\Delta X^{(t)}\Delta s^{(j-t)});$$

$$\mu(\alpha)=\frac{x(\alpha)^{\mathrm{T}}s(\alpha)}{n}=1-\frac{\alpha\|w\|}{n}+\frac{1}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{t=j-r}^{r}(\Delta x^{(t)})^{\mathrm{T}}\Delta s^{(j-t)}.$$

Proof We can come to the conclusion by brief calculation.

Lemma 2 *Suppose $(x,s)\in\Omega^+,\Delta w=(\Delta x^{(j)},\Delta s^{(j)})\,(j=1,2,\ldots,r)$ is the solution of systems (2),(3),(3') and let $D=(X^{-1}S)^{\frac{1}{2}}$,*

$$h^1=-(XS)^{-\frac{1}{2}}\frac{W^2e}{\|w\|}(k=1),\ h^{(k)}=(XS)^{-\frac{1}{2}}\{-\sum_{t=1}^{k-1}\Delta X^{(t)}\Delta s^{(k-t)}\}\ \ (k=2,\ldots,r).$$

Then for $k = 1, 2, \ldots, r$, we have

(i)

$$-\kappa \parallel h^{(k)} \parallel^2 \leq (\Delta x^{(k)})^{\mathrm{T}} \Delta s^{(k)} \leq \frac{1}{4} \parallel h^{(k)} \parallel^2; \tag{4}$$

(ii)

$$\max\{\parallel D\Delta x^{(k)} \parallel, \parallel D^{(-1)}\Delta s^{(k)} \parallel\} \leq \sqrt{2\kappa+1} \parallel h^{(k)} \parallel . \tag{5}$$

Proof (i) For $k = 1$, let $q^{(1)} = D\Delta x^{(1)} - D^{-1}\Delta s^{(1)}$. Taking norms, we have

$$\parallel D\Delta x^{(1)} \parallel^2 + \parallel D^{-1}\Delta s^{(1)} \parallel^2 - 2(\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} = \parallel q^{(1)} \parallel^2 .$$

Since $\parallel q^{(1)} \parallel^2 \geq 0$, we get

$$2(\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} \leq \parallel D\Delta x^{(1)} \parallel^2 + \parallel D^{-1}\Delta s^{(1)} \parallel^2 .$$

Multiplying both sides of (3) by $(Xs)^{-\frac{1}{2}}$, and then taking norms, we have

$$\parallel D\Delta x^{(1)} \parallel^2 + \parallel D^{-1}\Delta s^{(1)} \parallel^2 + 2(\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} = (h^1)^{\mathrm{T}} h^1.$$

Whence

$$(\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} \leq \frac{1}{4} \parallel h^1 \parallel^2 . \tag{6}$$

From $M\Delta x^{(1)} = \Delta s^{(1)}$ and the definition of $P_*(\kappa)$-matrix, we obtain

$$\begin{aligned}(\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} = (\Delta x^{(1)})^{\mathrm{T}} \cdot M\Delta x^{(1)} &\geq -4\kappa \sum_{i\in I}(\Delta x^{(1)})_i (M\Delta x^{(1)})_i \\ &= -4\kappa \sum_{i\in I}(\Delta x^{(1)})_i(\Delta s^{(1)})_i. \end{aligned} \tag{7}$$

Then $(D\Delta x^{(1)})_i + (D^{-1}\Delta s^{(1)})_i = (h^1)_i$ and we have

$$\begin{aligned}(\Delta x^{(1)})_i(\Delta s^{(1)})_i &= (D\Delta x^{(1)})_i(D^{-1}\Delta s^{(1)})_i = (D\Delta x^{(1)})_i[(h^1)_i - (D\Delta x^{(1)})_i] \\ &= -[(D\Delta x^{(1)})_i - \frac{1}{2}(h^1)_i]^2 + \frac{1}{4}((h^1)_i)^2 \leq \frac{1}{4} \parallel h^1 \parallel^2 . \end{aligned} \tag{8}$$

Combining (6),(7) and (8) yields $-\kappa \parallel h^1 \parallel^2 \leq (\Delta x^{(1)})^{\mathrm{T}}\Delta s^{(1)} \leq \frac{1}{4} \parallel h^1 \parallel^2$.

For $k = 2, \ldots, r$, the proof is similar.

(ii) For $k = 1$, it holds obviously. Now we prove on $k = 2, \ldots, r$. Since

$$\parallel D\Delta x^{(k)} \parallel^2 + \parallel D^{-1}\Delta s^{(k)} \parallel^2 + 2(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(k)} = \parallel h^{(k)} \parallel^2$$

and

$$-\kappa \parallel h^{(k)} \parallel^2 \leq (\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(k)} \leq \frac{1}{4} \parallel h^{(k)} \parallel^2,$$

we have

$$\parallel D\Delta x^{(k)} \parallel^2 + \parallel D^{-1}\Delta s^{(k)} \parallel^2 + 2(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(k)} \geq \parallel D\Delta x^{(k)} \parallel^2 + \parallel D^{-1}\Delta s^{(k)} \parallel^2 - 2\kappa \parallel h^{(k)} \parallel^2 .$$

Therefore,

$$\max\{\parallel D\Delta x^{(k)} \parallel, \parallel D^{-1}\Delta s^{(k)} \parallel\} \leq \sqrt{2\kappa+1} \parallel h^{(k)} \parallel .$$

This completes the proof. □

Lemma 3 *Let* $(x,s)\in N_\infty^-(\beta), \Delta w=(\Delta x^{(k)},\Delta s^{(k)})\,(k=1,2,\ldots,r)$ *is the solution of systems (2), (3), (3'). Then for* $k=1,2,\ldots,r$, *we have*

$$\max\{\| D\Delta x^{(k)} \|, \| D^{-1}\Delta s^{(k)} \|\} \le \frac{p(k)(2\kappa+1)^{\frac{2k-1}{2}}((h^1)^{\mathrm T}h^1)^{\frac{k}{2}}}{f_{\min}^{\frac{k-1}{2}}}, \tag{9}$$

where $p(k)$ *is defined recursively as:* $p(1)=1, p(k)=\sum_{i=1}^{k-1}p(i)p(k-i)$, $f_{\min}=\min\{x_is_i \mid i=1,2,\ldots,n\}$.

Proof The proof is by induction on k. Note that (9) trivially holds for $k=1$. Assume (9) holds for all i, with $1\le i<k$. We will show that it holds for k. For $(3')$. Taking $(XS)^{(-\frac{1}{2})}$, we have

$$D\Delta x^{(k)}+D^{-1}\Delta s^{(k)}=-(XS)^{-\frac{1}{2}}\sum_{i=1}^{k-1}\Delta X^{(i)}\Delta s^{(k-i)}.$$

From (5), we obtain

$$\max\{\| D\Delta x^{(k)} \|, \| D^{-1}\Delta s^{(k)} \|\} \le \sqrt{2\kappa+1}\,\| h^{(k)} \|. \tag{10}$$

On the other hand, we have by induction hypothesis

$$\begin{aligned}
\| (XS)^{-\frac{1}{2}}h^{(k)} \| &\le \frac{1}{f_{\min}^{\frac{1}{2}}}\sum_{i=1}^{k-1}\| D\Delta x^{(i)} \|\| D^{-1}\Delta s^{(k-i)} \| \\
&\le \frac{1}{f_{\min}^{\frac{1}{2}}}\sum_{i=1}^{k-1}\Big(\frac{p(i)(2\kappa+1)^{\frac{2i-1}{2}}((h^1)^{\mathrm T}h^1)^{\frac{i}{2}}}{f_{\min}^{\frac{i-1}{2}}}\Big)\Big(\frac{p(k-i)(2\kappa+1)^{\frac{2(k-i)-1}{2}}((h^1)^{\mathrm T}h^1)^{\frac{k-i}{2}}}{f_{\min}^{\frac{k-i-1}{2}}}\Big) \\
&= \frac{1}{f_{\min}^{\frac{k-1}{2}}}((h^1)^{\mathrm T}h^1)^{\frac{k}{2}}(2\kappa+1)^{\frac{2k-2}{2}}\sum_{t=1}^{k-1}p(t)p(k-t) \\
&= \frac{p(k)((h^1)^{\mathrm T}h^1)^{\frac{k}{2}}(2\kappa+1)^{\frac{2k-2}{2}}}{f_{\min}^{\frac{k-1}{2}}}.
\end{aligned}$$

This completes the proof. □

Lemma 4 *Suppose Lemma 3 holds. For* $j=r+1,\ldots,2r$, *we have*

$$\begin{aligned}
\| \sum_{k=j-r}^{r}\Delta X^{(k)}\Delta s^{(j-k)} \| &\le \frac{((h^1)^{\mathrm T}h^1)^{\frac{j}{2}}(2\kappa+1)^{j-1}}{(1-\beta)^{\frac{j}{2}-1}}\cdot\frac{16^r}{8r} \\
&\le \Big(\frac{n}{1-\beta}\Big)^{\frac{j}{2}}(2\kappa+1)^{j-1}\cdot\frac{16^r}{8r}\cdot(1-\beta).
\end{aligned} \tag{11}$$

Proof From Lemma 3, we have

$$\begin{aligned}
&\| \sum_{k=j-r}^{r}\Delta X^{(k)}\Delta s^{(j-k)} \| \le \sum_{k=j-r}^{r}\| D\Delta X^{(k)} \|\,\| D^{-1}\Delta s^{(j-k)} \| \\
&\le \sum_{k=j-r}^{r}\Big(\frac{p(k)(2\kappa+1)^{\frac{2k-1}{2}}((h^1)^{\mathrm T}h^1)^{\frac{k}{2}}}{(1-\beta)^{\frac{k-1}{2}}}\Big)\cdot\Big(\frac{p(j-k)(2\kappa+1)^{\frac{2(j-k)-1}{2}}((h^1)^{\mathrm T}h^1)^{\frac{j-k}{2}}}{(1-\beta)^{\frac{j-k-1}{2}}}\Big) \\
&= \frac{((h^1)^{\mathrm T}h^1)^{\frac{j}{2}}(1+2\kappa)^{j-1}}{(1-\beta)^{\frac{j}{2}-1}}\sum_{k=j-r}^{r}p(k)p(j-k)
\end{aligned}$$

$$= \frac{((h^1)^{\mathrm{T}}h^1)^{\frac{j}{2}}(1+2\kappa)^{j-1}}{(1-\beta)^{\frac{j}{2}-1}}(\sum_{k=1}^{r}p(k)p(j-k) - \sum_{k=1}^{j-r-1}p(k)p(j-k))$$

$$\leq \frac{((h^1)^{\mathrm{T}}h^1)^{\frac{j}{2}}(1+2\kappa)^{j-1}}{(1-\beta)^{\frac{j}{2}-1}}p(2r)$$

$$\leq \frac{((h^1)^{\mathrm{T}}h^1)^{\frac{j}{2}}(1+2\kappa)^{j-1}}{(1-\beta)^{\frac{j}{2}}} \cdot \frac{16^r}{8r} \cdot (1-\beta) \quad (p(j) \leq \frac{2^{2j-1}}{j})$$

$$= (\frac{n}{1-\beta})^{\frac{j}{2}}(2\kappa+1)^{j-1} \cdot \frac{16^r}{8r} \cdot (1-\beta) \quad (h^1)^{\mathrm{T}}h^1 = \| h^1 \|^2 \leq \frac{\| W^{\frac{3}{2}}e \|^2}{\| w \|^2} \leq \| w^{\frac{1}{2}} \|^2 = n$$

This completes the proof of the lemma. □

Lemma 5 *Let $\nu(\alpha) = 1 - \frac{\alpha\|w\|}{n}$, and $j = 1, 2, \ldots, n$, $Xs \geq (1-\beta)e$, we have*

$$\min_j\{(w_j - \frac{\alpha \| w \|}{n}) - (1-\beta)\nu(\alpha)\} \geq \frac{\alpha\beta(1-\beta)}{\sqrt{n}}.$$

Proof From Lemma 3.1 in [10], we know the lemma holds.

Theorem 1 *Let $(x, s) \in N_\infty^-(\beta)$, and $\alpha = n^{-\frac{1}{2r}} \cdot \frac{1-\beta}{16n} \cdot \sqrt[r]{4\beta}(2\kappa+1)^{-2}$. We have*

(i) $X(\alpha)s(\alpha) \geq (1-\beta)\mu(\alpha)e$;

(ii) $(x(\alpha), s(\alpha)) \in \Omega^+$.

Proof Let $\eta = \sum_{j=r+1}^{2r} \alpha^j \sum_{k=j-r}^{r} \Delta X^{(k)}\Delta s^{(j-k)}$. $(x, s) \in N_\infty^-(\beta)$ implies $Xs \geq (1-\beta)e$. By Lemma 4, we obtain

$$\| \eta \|_\infty \leq \| \eta \| \leq \sum_{j=r+1}^{2r} \alpha^j \cdot \| \sum_{k=j-r}^{r} \Delta X^{(k)}\Delta s^{(j-k)} \| \leq \sum_{j=r+1}^{2r} \alpha^j (\frac{n}{1-\beta})^{\frac{j}{2}} \cdot (1-\beta) \cdot (2\kappa+1)^{j-1} \cdot \frac{16^r}{8r}$$

$$\leq \sum_{j=r+1}^{2r} \alpha^j (\frac{n}{1-\beta})^{\frac{j}{2}} \cdot (1-\beta) \cdot (2\kappa+1)^{2r-1} \cdot \frac{16^r}{8r}$$

$$\leq r\alpha^{r+1} \cdot (\frac{n}{1-\beta})^r \cdot (1-\beta) \cdot (2\kappa+1)^{2r} \cdot \frac{16^r}{8r}$$

$$\leq r\alpha \cdot (n^{-\frac{1}{2r}} \cdot \frac{1-\beta}{16n} \cdot \sqrt[r]{4\beta}(2\kappa+1)^{-2})^r \cdot (\frac{n}{1-\beta})^r \cdot (1-\beta) \cdot (2\kappa+1)^{2r} \cdot \frac{16^r}{8r}$$

$$= \frac{1}{2}n^{-\frac{1}{2}} \cdot (1-\beta)\alpha\beta$$

$$\leq \frac{n}{n+1+\beta} \cdot n^{-\frac{1}{2}} \cdot (1-\beta)\alpha\beta \quad (n \geq 2). \tag{12}$$

Since $\frac{n}{n+1+\beta} = 1 - \frac{1+\beta}{n+1+\beta}$, we have

$$\| \eta \|_\infty \leq (1 - \frac{1+\beta}{n+1+\beta}) \cdot \frac{1}{\sqrt{n}} \cdot (1-\beta)\alpha\beta$$

$$\leq \frac{1}{\sqrt{n}} \cdot (1-\beta)\alpha\beta - \frac{1+\beta}{n+1+\beta} \cdot \frac{1}{\sqrt{n}} \cdot (1-\beta)\alpha\beta$$

$$= \frac{1}{\sqrt{n}} \cdot (1-\beta)\alpha\beta - \frac{1+\beta}{n} \cdot \frac{n}{n+1+\beta} \cdot \frac{1}{\sqrt{n}} \cdot (1-\beta)\alpha\beta$$

$$\le \frac{1}{\sqrt{n}}\cdot(1-\beta)\alpha\beta - \frac{1+\beta}{n}\cdot \parallel \eta \parallel_\infty \le \frac{1}{\sqrt{n}}\cdot(1-\beta)\alpha\beta - \frac{1+\beta}{n^2}\parallel \eta \parallel_\infty$$
$$\le \frac{1}{\sqrt{n}}\cdot(1-\beta)\alpha\beta - \frac{1+\beta}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{k=j-r}^{r}(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(j-k)}$$
$$\le \frac{1}{\sqrt{n}}\cdot(1-\beta)\alpha\beta - \frac{1-\beta}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{k=j-r}^{r}(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(j-k)}.$$

From Lemma 5, we have

$$\min_j\{(w_j-\frac{\alpha\parallel w\parallel}{n})-(1-\beta)\nu(\alpha)\}\ge\frac{\alpha\beta(1-\beta)}{\sqrt{n}}.$$

As a sequence

$$\parallel \eta \parallel_\infty \le w_j-\frac{\alpha w_j^2}{\parallel w\parallel}-(1-\beta)\nu(\alpha)-\frac{1-\beta}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{k=j-r}^{r}(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(j-k)}$$
$$=-(1-\beta)[\nu(\alpha)+\frac{1}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{k=j-r}^{r}(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(j-k)}]+w_j-\frac{\alpha w_j^2}{\parallel \omega\parallel}.$$

Thereby, $\eta_i+w_j-\frac{\alpha w_j^2}{\|w\|}\ge(1-\beta)\mu(\alpha)$. We have $X(\alpha)s(\alpha)\ge(1-\beta)\mu(\alpha)e$. The proof (ii) is similar to the theorem of [11].

Corollary 1 *Let* $(x_0,s_0)\in N_\infty^-(\beta)$, $\beta\in(0,1), r\in[1,n]$. *Then the algorithms generate a strict solution satisfying* $(x^k)^{\mathrm{T}}s^k\le\varepsilon$ *and at most* $K=[\frac{n^{\frac{1}{2r}+\frac{1}{2}}\cdot 16n\cdot(2\kappa+1)^2}{\beta(1-\beta)\sqrt[r]{4\beta}}\log\frac{x_0^{\mathrm{T}}s_0}{\varepsilon}]$.

Proof From $\mu(\alpha)=\frac{x(\alpha)^{\mathrm{T}}s(\alpha)}{n}=1-\frac{\alpha\|w\|}{n}+\frac{1}{n}\sum_{j=r+1}^{2r}\alpha^j\sum_{t=j-r}^{r}(\Delta x^{(t)})^{\mathrm{T}}\Delta s^{(j-t)}$, and

$$|(\Delta x^{(k)})^{\mathrm{T}}\Delta s^{(k)}|\le\sqrt{n}\parallel \Delta X^{(k)}\Delta s^{(k)}\parallel$$

and from (12), we have

$$\mu(\alpha)\le 1-\frac{\alpha\parallel w\parallel}{n}+\frac{\sqrt{n}}{n}\cdot\frac{n}{n+1+\beta}\cdot\frac{\alpha\beta(1-\beta)\mu}{\sqrt{n}}$$
$$=1-\frac{\alpha\parallel w\parallel}{n}+\frac{\alpha\beta(1-\beta)}{n+1+\beta}$$
$$\le 1-\frac{\alpha\beta}{\sqrt{n}}\quad(\parallel w\parallel\ge\sqrt{n}).$$

Then $(x^k)^{\mathrm{T}}s^k<n(1-\frac{\alpha\beta}{\sqrt{n}})^k\mu_0$. Taking logarithms, we have

$$\log(x^k)^{\mathrm{T}}s^k\le k\log[(1-\frac{\alpha\beta}{\sqrt{n}})]+\log(x_0)^{\mathrm{T}}s_0$$
$$\le -k\cdot\frac{\beta(1-\beta)}{n^{\frac{1}{2r}+\frac{1}{2}}\cdot 16n}\cdot\sqrt[r]{4\beta}\cdot(2\kappa+1)^{-2}+\log(x_0)^{\mathrm{T}}s_0.$$

Hence, when $k\ge K$,we have $\log(x^k)^{\mathrm{T}}s^k\le\log\varepsilon$, that is to say $(x^k)^{\mathrm{T}}s^k\le\varepsilon$.

4. Computational experiments

To verify that our algorithm has better practicality than [12], we present two numerical experiments by using a Matlab code. For given $\varepsilon = 10^{-6}, \kappa = 0.25$, $\beta = 0.5, r = 8$, $q = (-1, \dots, -1)^{\mathrm{T}}$. In the following tables, we denote by t_1, t_2 iterations of our algorithms and [12], respectively. The results show the validity of our algorithm.

Murty Example[13] This is a linear complementarity problem. The solution is $x^* = (0, \dots, 0, 1)^{\mathrm{T}}$, $y^* = (1, \dots, 1, 0)^{\mathrm{T}}$.

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 1 & 2 & \cdots & 2 & 2 \\ 0 & 0 & 1 & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For $n = 8$, we get the initial points by programming:

$$x_0 = (0.65037,\ 0.65033,\ 0.65016,\ 0.65032,\ 0.6503,\ 0.6503,\ 0.65031,\ 2.1163),$$

$$s_0 = (11.686,\ 10.386,\ 9.0852,\ 7.7847,\ 6.4841,\ 5.1835,\ 3.8829,\ 1.1163).$$

Stopping at 31th gives the solution

$$x^* = (0.00018, 0.00054, 0.00032, 0.00047, 0.00039, 0.00017, 0.00019, 0.97462),$$

$$s^* = (0.91876, 0.97825, 0.96582, 0.85473, 0.76042, 0.75047, 0.65932, 0.00021).$$

For various dimensions n, we get the following results:

n	8	16	32	64	128	256
t_1	31	56	78	99	122	145
t_2	67	132	157	189	256	301

Table 1 Murty problem

Ahn Example[14] This is a linear complementarity problem. The solution is $x^* = (1, 0, \dots, 0)^{\mathrm{T}}$, $y^* = (0, 1, \dots, 1)^{\mathrm{T}}$.

$$M = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 4 & -2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 4 \end{pmatrix}.$$

We choose the initial points:

$$x_0 = (1.0106,\ 0.99505,\ 0.99891,\ 0.99899,\ 0.99948,\ 0.99934,\ 0.99917,\ 0.99935),$$

$$s_0 = (1.0532,\ 1.993,\ 1.9927,\ 1.9959,\ 1.9982,\ 1.9985,\ 1.9973,\ 3.9966).$$

When $n = 8$, stopping at 35th yields the solution:

$$x^* = (1.57821,\ 1.29372,\ 1.27689,\ 1.17186,\ 1.20712,\ 1.10475,\ 0.98793,\ 0.65472),$$

$$s^* = (0.00004,\ 0.00012,\ 0.00035,\ 0.00047,\ 0.00052,\ 0.00068,\ 0.00003,\ 0.00021).$$

For various dimensions n, we get the following results:

n	8	16	32	64	128	256
t_1	35	51	79	109	136	169
t_2	75	112	190	245	278	314

Table 2 Ahn problem

References

[1] KARMARKAR N. *A new polynomial-time algorithm for linear programming* [J]. Combinatorica, 1984, **4**(4): 373–395.

[2] YE Yinyu. *Interior Point Algorithms Theory and Analysis* [M]. John Wiley & Sons, Inc., New York, 1997.

[3] TODD M J, YE Yinyu. *A centered projective algorithm for linear programming* [J]. Math. Oper. Res., 1990, **15**(3): 508–529.

[4] VANDERBEI R J, MEKETON M S, FREEDMAN B A. *A modification of Karmarkar's linear programming algorithm* [J]. Algorithmica, 1986, **1**(4): 395–407.

[5] DIKIN I I. *Iterative solution of problems of linear and quadratic programming* [J]. Dokl. Akad. Nauk SSSR, 1967, **174**: 747–748. (in Russian)

[6] GONZAGA C C. *Path-following methods for linear programming* [J]. SIAM Rev., 1992, **34**(2): 167–224.

[7] GÜLER O. *Existence of interior points and interior paths in nonlinear monotone complementarity problems* [J]. Math. Oper. Res., 1993, **18**(1): 128–147.

[8] HUNG Pifang, YE Yinyu. *An asymptotical $O(\sqrt{n}L)$-iteration path-following linear programming algorithm that uses wide neighborhoods* [J]. SIAM J. Optim., 1996, **6**(3): 570–586.

[9] POTRA F A. *A superlinearly convergent predictor-corrector method for degenerate LCP in a wide neighborhood of the central path with $O(\sqrt{n}L)$-iteration complexity* [J]. Math. Program., Ser.A, 2004, **100**(2): 317–337.

[10] JANSEN B, ROOS C, TERLAKY T. et al. *Improved complexity using higher-order correctors for primal-dual Dikin affine scaling* [J]. Math. Programming, Ser.B, 1997, **76**(1): 117–130.

[11] ZHANG Mingwang, HUANG Chongchao. *A higher-order affine scaling algorithm for a class of nonmonotonic linear complementary problems* [J]. Math. Numer. Sin., 2004, **26**(1): 37–46. (in Chinese)

[12] ZHANG Mingwang. *A higher-order Dikin-type affine scaling algorithm for $P_*(\kappa)$ linear complementary problems* [J]. J. Lanzhou Univ. Technol., 2006, **32**(3): 141–144. (in Chinese)

[13] MURTY K G. *Linear and Nonlinear Programming* [M]. Helderman, Berlin, 1994.

[14] KANZOW C. *Global convergence properties of some iterative methods for linear complementarity problems* [J]. SIAM J. Optim., 1996, **6**(2): 326–341.