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One Kind of Complete Lie Algebra over a Commutative Ring

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Abstract In this paper, we obtain a new kind of complete Lie algebra over a commutative ring, which is the Lie algebra consisting of all $n \times n$ anti-symmetric matrices over a 2-torsionfree commutative ring with identity.

Keywords complete Lie algebra; anti-symmetric matrices; commutative ring.

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1. Introduction

Since 1980's, system research has been done in studying the completeness of Lie algebras over fields[1−3]. However, the study of the complete Lie algebras over commutative rings is little done. We know that the completeness of the Lie algebras is closely related to the derivations of the Lie algebras. Recently, many papers have been concerned with the study of derivations or automorphisms of matrix algebras (or matrix Lie algebras) over commutative rings. Wang[4] determined the derivations of any Lie subalgebras of the general linear Lie algebra containing upper triangular matrix algebra. Wang[5] gave an explicit description on the derivations of any Lie subalgebras of upper triangular matrix algebra containing diagonal matrix algebra. Huang[6] studied the Jordan isomorphisms of the algebra consisting of all $n \times n$ symmetric matrices over a commutative principal ideal domain. We know that the set of all $n \times n$ symmetric matrices forms a Jordan algebra with addition of matrices and the symmetrized multiplication $A \circ B = AB + BA$, and the set of all $n \times n$ anti-symmetric matrices (matrix A satisfies $A = -A'$, where A' is the transpose of A) forms a Lie algebra with the bracket operation $[A, B] = AB - BA$. Inspired by this, we intend to investigate the completeness of this Lie algebra.

2. Preliminaries

Let R be a 2-torsionfree commutative ring with identity and $\mathrm{gl}(n, R)$ the general linear Lie algebra consisting of all $n \times n$ matrices over R with the bracket operation $[A, B] = AB - BA$ for any A,B $\in \mathrm{gl}(n, R)$. Denote by $\mathcal{L}_n(R)$ the Lie subalgebra of $\mathrm{gl}(n, R)$ consisting of all $n \times n$

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anti-symmetric matrices over R. Let E denote the identity matrix in $\mathrm{gl}(n, R)$, and $E_{i,j}$ the matrix with 1 at the position (i, j) and zeros elsewhere. Set $T_{i,j} = E_{i,j} - E_{j,i}$ for $i \neq j$. Then the matrix set $\{T_{i,j} | 1 \leq i < j \leq n\}$ is a basis of $\mathcal{L}_n(R)$. For any $x \in \mathcal{L}_n(R)$, $x = \sum_{1\leq i<j\leq n} a_{ij}T_{ij}$ for some $a_{ij} \in R$.

Lemma 2.1 *The set $\{T_{i,i+1} \mid i = 1, \ldots, n-1\}$ generates $\mathcal{L}_n(R)$ by Lie bracket operation.*

Proof The proof is trivial, thus omitted. □

A Lie algebra is called complete if its center is trivial and all its derivations are inner.

Let ϕ be a derivation of $\mathcal{L}_n(R)$. Lemma 2.1 implies that the set $\{\phi(T_{i,i+1}) \mid i = 1, \ldots, n-1\}$ generates $\mathcal{L}_n(R)$. So we will investigate $\phi(T_{i,i+1})$ $(i = 1, \ldots, n-1)$ and we write $\phi(T_{i,i+1})$ as

$$\phi(T_{i,i+1}) = \sum_{1\leq k<l\leq n} a_{kl}^{(i)} T_{kl}, \quad \text{where } a_{kl}^{(i)} \in R. \tag{2.1}$$

3. The completeness of $\mathcal{L}_n(R)$

Before giving the main result of this paper, we first give a useful lemma, which will be used later.

Lemma 3.1 *Let ϕ be a derivation of $\mathcal{L}_n(R)$ and $n \geq 5$. If $\phi(T_{i,i+1})$ $(i = 1, \ldots, n-1)$ are expressed as the forms of (2.1), then*

$$\begin{aligned}
\phi(T_{12}) &= \sum_{k=2}^{n} a_{1k}^{(1)} T_{1k} + \sum_{k=3}^{n} a_{2k}^{(1)} T_{2k}, \\
\phi(T_{i,i+1}) &= \sum_{k=i+1}^{n} a_{ik}^{(i)} T_{ik} + \sum_{k=i+2}^{n} a_{i+1,k}^{(i)} T_{i+1,k} + \sum_{k=1}^{i-1} a_{ki}^{(i)} T_{ki} + \sum_{k=1}^{i-1} a_{k,i+1}^{(i)} T_{k,i+1}, \\
&\quad \text{for } 2 \leq i \leq n-2, \\
\phi(T_{n-1,n}) &= \sum_{k=1}^{n-2} a_{k,n-1}^{(n-1)} T_{k,n-1} + \sum_{k=1}^{n-1} a_{kn}^{(n-1)} T_{kn},
\end{aligned}$$

and

$$a_{i,j+1}^{(i)} = a_{i+1,j}^{(j)}, \; a_{i+1,j}^{(i)} = a_{i,j+1}^{(j)}, \; a_{ij}^{(i)} = -a_{i+1,j+1}^{(j)}, \; a_{i+1,j+1}^{(i)} = -a_{ij}^{(j)} \text{ for } i+1 < j. \tag{3.1}$$

Proof Since $[T_{i,i+1}, T_{j,j+1}] = 0$ for $1 \leq i, j \leq n-1$ and $j \neq i-1, i, i+1$, we have, by applying ϕ, that

$$[\phi(T_{i,i+1}), T_{j,j+1}] + [T_{i,i+1}, \phi(T_{j,j+1})] = 0 \tag{3.2}$$

which results in that $a_{jk}^{(i)} = a_{j+1,k}^{(i)} = 0$ (in this equality, when $k > j$ or $k > j+1$, we assume that $a_{jk}^{(i)} = -a_{kj}^{(i)}$, $a_{j+1,k}^{(i)} = -a_{k,j+1}^{(i)}$) for $1 \leq j \leq n-1, j \neq i, i+1$, and $1 \leq k \leq n, k \neq i, i+1, j, j+1$. Now we get

$$\phi(T_{12}) = \sum_{k=2}^{n} a_{1k}^{(1)} T_{1k} + \sum_{k=3}^{n} a_{2k}^{(1)} T_{2k},$$

$$\phi(T_{n-1,n}) = \sum_{k=1}^{n-2} a_{k,n-1}^{(n-1)} T_{k,n-1} + \sum_{k=1}^{n-1} a_{kn}^{(n-1)} T_{kn},$$

$$\phi(T_{i,i+1}) = \sum_{k=i+1}^{n} a_{ik}^{(i)} T_{ik} + \sum_{k=i+2}^{n} a_{i+1,k}^{(i)} T_{i+1,k} + \sum_{k=1}^{i-1} a_{ki}^{(i)} T_{ki} + \sum_{k=1}^{i-1} a_{k,i+1}^{(i)} T_{k,i+1}$$

for $2 \le i \le n-2$ and $i \ne 3, n-3$, and

$$\phi(T_{34}) = \sum_{k=4}^{n} a_{3k}^{(3)} T_{3k} + \sum_{k=5}^{n} a_{4k}^{(3)} T_{4k} + \sum_{k=1}^{2} a_{k3}^{(3)} T_{k3} + \sum_{k=1}^{2} a_{k4}^{(3)} T_{k4} + a_{12}^{(3)} T_{12},$$

$$\phi(T_{n-3,n-2}) = \sum_{k=n-2}^{n} a_{n-3,k}^{(n-3)} T_{n-3,k} + \sum_{k=n-1}^{n} a_{n-2,k}^{(n-3)} T_{n-2,k} + \sum_{k=1}^{n-4} a_{k,n-3}^{(n-3)} T_{k,n-3} + \sum_{k=1}^{n-4} a_{k,n-2}^{(n-3)} T_{k,n-2} + a_{n,n-1}^{(n-3)} T_{n,n-1}.$$

In the following, we will prove $a_{12}^{(3)} = a_{n-1,n}^{(n-3)} = 0$.

From $[T_{34}, T_{25}] = 0$, we have $[\phi(T_{34}), T_{25}] + [T_{34}, \phi(T_{25})] = 0$. So we get $a_{12}^{(3)} = 0$. By $[T_{n-4,n-1}, T_{n-3,n-2}] = 0$, we get $[\phi(T_{n-4,n-1}), T_{n-3,n-2}] + [T_{n-4,n-1}, \phi(T_{n-3,n-2})] = 0$. Thus we have $a_{n-1,n}^{(n-3)} = 0$.

Also from (3.2) we have

$$a_{i,j+1}^{(i)} = a_{i+1,j}^{(j)}, \quad a_{i+1,j}^{(i)} = a_{i,j+1}^{(j)}, \quad a_{ij}^{(i)} = -a_{i+1,j+1}^{(j)}, \quad a_{i+1,j+1}^{(i)} = -a_{ij}^{(j)}.$$

The proof is completed. □

Theorem 3.2 *Let R be a 2-torsionfree commutative ring with identity and $n \ge 3$. Then $\mathcal{L}_n(R)$ is a complete Lie algebra.*

Proof Firstly, we will prove that the center of $\mathcal{L}_n(R)$ is trivial.

Let x be any element in the center of $\mathcal{L}_n(R)$. Since $[x, T_{i,i+1}] = 0, i = 1, \ldots, n-1$, we get $x = 0$.

Then, for any derivation ϕ of $\mathcal{L}_n(R)$, we will distinguish two cases to prove that ϕ is an inner derivation. Express $\phi(T_{i,i+1})$ $(i = 1, 2 \ldots, n-1)$ as the forms of (2.1).

Case 1 $n \ge 5$.

In this case, the proof will be given by steps.

Step 1. There exists an inner derivation ad x such that $(\phi - \text{ad } x)(T_{i,i+1}) = a_{i,i+1}^{(i)} T_{i,i+1}$.

In order to achieve our goal, we first do some preparing as follows.

Suppose that $\phi(T_{35}) = \sum_{1 \le i < j \le n} b_{ij} T_{ij}$ and $\phi(T_{14}) = \sum_{1 \le i < j \le n} c_{ij} T_{ij}$. Since $[T_{12}, T_{35}] = 0, [T_{23}, T_{14}] = 0$, and $[T_{35}, T_{14}] = 0$, by applying ϕ on the two sides of the above equalities, we have

$$[\phi(T_{12}), T_{35}] + [T_{12}, \phi(T_{35})] = 0,$$

$$[\phi(T_{23}), T_{14}] + [T_{23}, \phi(T_{14})] = 0,$$

$$[\phi(T_{35}), T_{14}] + [T_{35}, \phi(T_{14})] = 0.$$

From those equalities, we get $a_{23}^{(1)} = b_{15}, a_{12}^{(2)} = c_{34}, b_{15} = -c_{34}$. Thus

$$a_{23}^{(1)} = -a_{12}^{(2)}. \tag{3.3}$$

Suppose $\phi(T_{n-3,n}) = \sum_{1\le i<j\le n} d_{ij}T_{ij}$, by applying ϕ on $[T_{n-2,n-1}, T_{n-3,n}] = 0$, we get

$$[\phi(T_{n-2,n-1}), T_{n-3,n}] + [T_{n-2,n-1}, \phi(T_{n-3,n})] = 0,$$

which leads to $a_{n-1,n}^{(n-2)} = d_{n-3,n-2}$. By operating ϕ on $[T_{n-3,n-2}, T_{n-2,n}] = T_{n-3,n}$, we have

$$[\phi(T_{n-3,n-2}), T_{n-2,n}] + [T_{n-3,n-2}, \phi(T_{n-2,n})] = \phi(T_{n-3,n}).$$

This leads to $a_{n-3,n}^{(n-3)} = -d_{n-3,n-2}$. So

$$a_{n-1,n}^{(n-2)} = -a_{n-3,n}^{(n-3)}. \tag{3.4}$$

Let

$$x = a_{13}^{(2)}T_{12} + \sum_{i=3}^{n} a_{2i}^{(1)}T_{1i} - \sum_{j=1}^{n-2}\sum_{i=j+2}^{n} a_{ji}^{(j)}T_{j+1,i}.$$

By Lemma 3.1 and equalities (3.1), (3.3) and (3.4) a direct calculation shows that

$$(\phi - \mathrm{ad}\ x)(T_{i,i+1}) = a_{i,i+1}^{(i)}T_{i,i+1}.$$

Step 2. In this step, we intend to prove that $a_{i,i+1}^{(i)} = 0$ for $i = 1, 2, \ldots, n-1$.

Denote by ϕ_1 the $\phi - \mathrm{ad}\ x$, by applying ϕ_1 on the two sides of $[T_{i,i+1}, T_{i+1,i+2}] = T_{i,i+2}$, we get

$$\phi_1(T_{i,i+2}) = (a_{i,i+1}^{(i)} + a_{i+1,i+2}^{(i+1)})T_{i,i+2}.$$

By operating ϕ_1 on $[T_{i,i+1}, T_{i,i+2}] = -T_{i+1,i+2}$, we have

$$-(a_{i,i+1}^{(i)} + a_{i,i+1}^{(i)} + a_{i+1,i+2}^{(i+1)})T_{i+1,i+2} = -a_{i+1,i+2}^{(i+1)}T_{i+1,i+2}.$$

This implies

$$a_{i,i+1}^{(i)} + a_{i,i+1}^{(i)} + a_{i+1,i+2}^{(i+1)} = a_{i+1,i+2}^{(i+1)}.$$

So $a_{i,i+1}^{(i)} = 0$ for $1 \le i \le n-2$.

From $[T_{n-2,n}, T_{n-1,n}] = -T_{n-2,n-1}$, we have $a_{n-2,n-1}^{(n-2)} + a_{n-1,n}^{(n-1)} + a_{n-1,n}^{(n-1)} = a_{n-2,n-1}^{(n-2)}$. This implies $a_{n-1,n}^{(n-1)} = 0$.

Above discussion shows that $(\phi - \mathrm{ad}\ x)(T_{i,i+1}) = 0$ for $i = 1, 2, \ldots, n-1$. Since $T_{i,i+1}, i = 1, 2, \ldots, n-1$, generate $\mathcal{L}_n(R)$, we get $\phi = \mathrm{ad}\ x$ for $n \ge 5$.

Case 2 $3 \le n \le 4$.

Suppose $\phi(T_{i,i+2}) = \sum_{1\le k<l\le n} b_{kl}^{(i)}T_{kl}$. By applying ϕ on the two sides of $[T_{i,i+1}, T_{i+1,i+2}] = T_{i,i+2}$, $[T_{i,i+1}, T_{i,i+2}] = -T_{i+1,i+2}$ and $[T_{i,i+2}, T_{i+1,i+2}] = -T_{i,i+1}, i = 1, \ldots, n-2$, respectively, we get

$$[\phi(T_{i,i+1}), T_{i+1,i+2}] + [T_{i,i+1}, \phi(T_{i+1,i+2})] = \phi(T_{i,i+2}), \tag{3.5}$$

$$[\phi(T_{i,i+1}), T_{i,i+2}] + [T_{i,i+1}, \phi(T_{i,i+2})] = -\phi(T_{i+1,i+2}), \tag{3.6}$$

$$[\phi(T_{i,i+2}), T_{i+1,i+2}] + [T_{i,i+2}, \phi(T_{i+1,i+2})] = -\phi(T_{i,i+1}). \tag{3.7}$$

These lead to

$$\begin{gathered} a_{i,i+1}^{(i)} + a_{i+1,i+2}^{(i+1)} - b_{i,i+2}^{(i)} = 0, \\ a_{i,i+1}^{(i)} - a_{i+1,i+2}^{(i+1)} + b_{i,i+2}^{(i)} = 0, \text{ and } a_{i+1,i+2}^{(i)} = -a_{i,i+1}^{(i+1)}, \\ a_{i,i+1}^{(i)} - a_{i+1,i+2}^{(i+1)} - b_{i,i+2}^{(i)} = 0. \end{gathered} \tag{3.8}$$

So

$$a_{i,i+1}^{(i)} = 0, i = 1, \ldots, n-1. \tag{3.9}$$

When $n = 4$, from (3.5), (3.6) and (3.7) we also get

$$a_{14}^{(1)} = -a_{34}^{(2)}, \tag{3.10}$$

and

$$\begin{gathered} a_{34}^{(1)} + a_{14}^{(2)} + b_{24}^{(1)} = 0, \quad -a_{34}^{(1)} + a_{14}^{(2)} + b_{24}^{(1)} = 0, \\ a_{34}^{(1)} - a_{14}^{(2)} + b_{24}^{(1)} = 0, \quad a_{14}^{(2)} + a_{12}^{(3)} + b_{13}^{(2)} = 0, \\ -a_{14}^{(2)} + a_{12}^{(3)} + b_{13}^{(2)} = 0, \quad a_{14}^{(2)} - a_{12}^{(3)} + b_{13}^{(2)} = 0. \end{gathered}$$

From those equalities we get

$$a_{34}^{(1)} = a_{14}^{(2)} = a_{12}^{(3)} = 0. \tag{3.11}$$

By applying ϕ on $[T_{12}, T_{34}] = 0$, we have

$$[\phi(T_{12}), T_{34}] + [T_{12}, \phi(T_{34})] = 0.$$

It follows that

$$a_{24}^{(1)} = -a_{13}^{(3)}, \ a_{23}^{(1)} = a_{14}^{(3)}, \ a_{14}^{(1)} = a_{23}^{(3)}, \ a_{13}^{(1)} = -a_{24}^{(3)}. \tag{3.12}$$

Let

$$x = a_{13}^{(2)} T_{12} + \sum_{i=3}^{n} a_{2i}^{(1)} T_{1i} - \sum_{j=1}^{n-2} \sum_{i=j+2}^{n} a_{ji}^{(j)} T_{j+1,i}.$$

When $n = 3$, a direct calculation shows that $(\phi - \text{ad } x)(T_{i,i+1}) = a_{i,i+1}^{(i)} T_{i,i+1} = 0$ for $i = 1, 2$ due to (3.8) and (3.9).

For $n = 4$, note the equalities (3.8)–(3.12), then a direct calculation shows that

$$\begin{aligned} (\phi - \text{ad } x)(T_{12}) &= a_{12}^{(1)} T_{12} + a_{34}^{(1)} T_{34} = 0, \\ (\phi - \text{ad } x)(T_{23}) &= a_{23}^{(2)} T_{23} + a_{14}^{(2)} T_{14} = 0, \\ (\phi - \text{ad } x)(T_{34}) &= a_{12}^{(3)} T_{12} + a_{34}^{(3)} T_{34} = 0. \end{aligned}$$

Now we get the conclusion that $\phi = \text{ad } x$ for $3 \leq n \leq 4$. The proof is completed. □

Remark 1 When $n = 2$, for any derivation ϕ of $\mathcal{L}_2(R)$, we have $\phi(T_{12}) = aT_{12}$ for some $a \in R$. It is clear that it is a derivation but not an inner derivation. So $\mathcal{L}_2(R)$ is not a complete Lie algebra.

Remark 2 $\mathcal{L}_n(R)$ is a non-solvable Lie algebra when $n \geq 3$. Since for any ideal L of R, $\mathcal{L}_n(L)$

is an ideal of $\mathcal{L}_n(R)$, we know that $\mathcal{L}_n(R)$ is not simple. If R has an ideal L_1 such that $x_1x_2 = 0$ for any $x_1, x_2 \in L_1$ (we will give an example later), then $\mathcal{L}_n(L_1)$ is a solvable ideal of $\mathcal{L}_n(R)$. So $\mathcal{L}_n(R)$ is not semisimple.

Example Let $R = Z_4$, $L_1 = \{\bar{0}, \bar{2}\}$. Then L_1 is an ideal of R such that $x_1x_2 = 0$ for any $x_1, x_2 \in L_1$.

References

[1] MENG Daoji. *Some results on complete Lie algebras* [J]. Comm. Algebra, 1994, **22**(13): 5457–5507.

[2] MENG D J, WANG S P. *On the construction of complete Lie algebras* [J]. J. Algebra, 1995, **176**(2): 621–637.

[3] ZHU Linsheng, MENG Daoji. *One kind of complete Lie algebras* [J]. Algebras Groups Geom., 2000, **17**(1): 57–71.

[4] WANG Dengyin, YU Qiu. *Derivations of the parabolic subalgebras of the general linear Lie algebra over a commutative ring* [J]. Linear Algebra Appl., 2006, **418**(2-3): 763–774.

[5] WANG Dengyin, OU Shikun, YU Qiu. *Derivations of the intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices over a commutative ring* [J]. Linear Multilinear Algebra, 2006, **54**(5): 369–377.

[6] HUANG Liping, BAN Tao, LI Deqiong. et al. *Jordan isomorphisms and additive rank preserving maps on symmetric matrices over PID* [J]. Linear Algebra Appl., 2006, **419**(2-3): 311–325.