Journal of Mathematical Research & Exposition
Jan., 2009, Vol. 29, No. 1, pp. 91–98
DOI:10.3770/j.issn:1000-341X.2009.01.012
Http://jmre.dlut.edu.cn

$L(2,1)$-Circular Labelings of Cartesian Products of Complete Graphs

LÜ Da Mei[1], *LIN Wen Song*[2], *SONG Zeng Min*[2]
(1. Department of Mathematics, Nantong University, Jiangsu 226001, China;
2. Department of Mathematics, Southeast University, Jiangsu 210096, China)
(E-mail: damei@ntu.edu.cn)

Abstract For positive integers j and k with $j \geq k$, an $L(j,k)$-labeling of a graph G is an assignment of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least j, and the difference between labels of vertices that are distance two apart is at least k. The span of an $L(j,k)$-labeling of a graph G is the difference between the maximum and minimum integers it uses. The $\lambda_{j,k}$-number of G is the minimum span taken over all $L(j,k)$-labelings of G. An m-(j,k)-circular labeling of a graph G is a function $f: V(G) \to \{0,1,2,\ldots,m-1\}$ such that $|f(u)-f(v)|_m \geq j$ if u and v are adjacent; and $|f(u)-f(v)|_m \geq k$ if u and v are at distance two, where $|x|_m = \min\{|x|, m-|x|\}$. The minimum integer m such that there exists an m-(j,k)-circular labeling of G is called the $\sigma_{j,k}$-number of G and is denoted by $\sigma_{j,k}(G)$. This paper determines the $\sigma_{2,1}$-number of the Cartesian product of any three complete graphs.

Keywords $\lambda_{2,1}$-number; $\sigma_{2,1}$-number; Cartesian product.

Document code A
MR(2000) Subject Classification 05C17
Chinese Library Classification O157.5

1. Introduction

For two positive integers j and k with $j \geq k$, an $L(j,k)$-labeling of a graph G is an assignment L of nonnegative integers to $V(G)$ such that the difference between labels of adjacent vertices is at least j, and the difference between labels of vertices that are distance two apart is at least k. Elements of the image of L are called labels, and the span of L, denoted by span(L), is the difference between the largest and smallest labels of L. The $\lambda_{j,k}$-number of G, denoted $\lambda_{j,k}(G)$, is the minimum span over all $L(j,k)$-labelings of G. If L is an $L(j,k)$-labeling with span $\lambda_{j,k}(G)$, then L is called a $\lambda_{j,k}$-labeling of G. We shall assume without loss of generality that the minimum label of $L(j,k)$-labelings of G is always 0. An m-(j,k)-circular labeling of a graph G is a function $f: V(G) \to \{0,1,2,\ldots,m-1\}$ such that $|f(u)-f(v)|_m \geq j$ if u and v are adjacent; and $|f(u)-f(v)|_m \geq k$ if u and v are at distance two, where $|x|_m = \min\{|x|, m-|x|\}$. The minimum integer m such that there exists an m-(j,k)-circular labeling of G is called the $\sigma_{j,k}$-number of G and is denoted by $\sigma_{j,k}(G)$.

Received date: 2006-12-17; **Accepted date**: 2007-09-04
Foundation item: the National Natural Science Foundation of China (No. 10671033); the Science Foundation of Southeast University (No. XJ0607230); the Natural Science Foundation of Nantong University (No. 08Z003).

Motivated by a special kind of channel assignment problem, Griggs and Yeh[7] first proposed and studied the $L(2,1)$-labeling of a graph. Since then the $\lambda_{2,1}$-numbers of graphs have been studied extensively[1,3,5−7,9,12,14]. And $L(j,k)$-labelings were also investigated in many papers[2−5].

Given two graphs G and H, the Cartesian product of G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$ in which two vertices (x,y) and (x',y') are adjacent if $x = x'$ and $yy' \in E(H)$ or $y = y'$ and $xx' \in E(G)$. Let G^k denote the Cartesian product of k copies of G. Let K_n denote the complete graph on n vertices. Then $K_n^2 = K_n \times K_n$ and $K_n^3 = K_n \times K_n \times K_n$.

The $L(2,1)$-labeling of the Cartesian product of n paths, especially of the Cartesian product of n copies of P_2 (the n-cube Q_n), was investigated by Whittlesey, Georges, and Mauro[14]. In the same paper, they completely determined the $\lambda_{2,1}$-numbers of Cartesian products of two paths. Jha et al.[9] studied the $L(2,1)$-labeling of the Cartesian product of a cycle and a path. The $\lambda_{2,1}$-numbers of the Cartesian product of a cycle and a path were completely computed by Klavžar and Vesel in [10]. Partial results for the $\lambda_{2,1}$-numbers of the Cartesian products of two cycles were obtained in [10]. These partial results are completed in [13]. Georges, Mauro, and Whittlesey[6] determined $L(2,1)$-labeling numbers of Cartesian products of two complete graphs. This result was then extended by Georges, Mauro, and Stein[5] who determined the $L(j,k)$-labeling numbers of Cartesian products of two complete graphs.

Theorem 1.1[5] *Let j, k,n, and m be integers where $2 \le m < n$ and $j \ge k$. Then*

(i) $\lambda_{j,k}(K_n \times K_m) = (n-1)j + (m-1)k$, *if* $j/k > m$;

(ii) $\lambda_{j,k}(K_n \times K_m) = (nm-1)k$, *if* $j/k \le m$.

Theorem 1.2[5] *Let j, k, and n be integers where $2 \le n$ and $j \ge k$. Then*

(i) $\lambda_{j,k}(K_n^2) = (n-1)j + (2n-2)k$, *if* $j/k > n-1$;

(ii) $\lambda_{j,k}(K_n^2) = (n^2-1)k$, *if* $j/k \le n-1$.

Georges, and Mauro[3] also obtained other results on $L(j,k)$-labelling numbers of Cartesian products of complete graphs. In particular, they investigated the $\lambda_{j,k}$-number of K_n^3.

Theorem 1.3[3] *The $\lambda_{j,k}$-number of $Q_3 \cong K_2^3$ is equal to $3j$ if $j/k \le 5/2$; and $j + 5k$ if $j/k \ge 5/2$.*

Theorem 1.4[3] *Suppose n is an odd integer, $n \ge 3$. Then*

(i) $\lambda_{j,k}(K_n^3) = (n-1)(j+3k)$, *if* $j/k \ge 3n-4$;

(ii) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \le n-2$;

(iii) $\lambda_{j,k}(K_n^3) \le (n-1)(j+3k)$, *if* $n-2 < j/k < 3n-4$.

Theorem 1.5[3] *Suppose n is an even integer. Then*

(i) $\lambda_{j,k}(K_n^3) = (n^2-1)k$, *if* $j/k \le n/2$;

(ii) $$\lambda_{j,k}(K_n^3) \le \begin{cases} (n^2+2n)k, & \text{if } n/2 < j/k \le n-2, \\ n(j+3k), & \text{if } n-2 < j/k \le 2n(n-2), \\ (n-1)j + n(2n-1)k, & \text{if } j/k > 2n(n-2)). \end{cases}$$

Heuvel, Leese and Shepherd[8] first introduced $\sigma_{j,k}$-number of graphs, where infinite lattices were focused on. The following theorem is useful in the proof of our main result.

Theorem 1.6[8] *For any graph G, $\lambda_{2,1}(G)+1 \le \sigma_{2,1}(G) \le \lambda_{2,1}(G)+2$.*

Theorem 1.7[15] *For $n,m \ge 2$, then*

$$\sigma_{2,1}(K_n \times K_m) = \begin{cases} \lambda_{2,1}(K_2^2)+2=6, & \text{if } m=n=2, \\ \lambda_{2,1}(K_n \times K_m)+1=nm, & \text{otherwise.} \end{cases}$$

Theorem 1.8[16] *Let j, k, m and n be positive integers with $2 \le m < n$ and $j \ge k$. Then*

$$\sigma_{j,k}(K_n \times K_m) = \begin{cases} nmk, & \text{if } j/k \le m, \\ mj, & \text{if } j/k > m. \end{cases}$$

Theorem 1.9[16] *Let j, k, and n be positive integers with $n \ge 2$ and $j \ge k$. Then*

$$\sigma_{j,k}(K_n^2) = \begin{cases} n^2k, & \text{if } j/k \le n-1, \\ n(j+k), & \text{if } j/k > n-1. \end{cases}$$

Theorem 1.10[17] *Let n, m and l be positive integers with $n \ge m \ge l \ge 2$. If $n \ge 4$, then $\lambda_{2,1}(K_n \times K_m \times K_l) = nm-1$.*

Theorem 1.11[17]

$$\lambda_{2,1}(K_3 \times K_3 \times K_l) = \begin{cases} 9, & \text{if } l=2, \\ 10, & \text{if } l=3. \end{cases}$$

By Theorem 1.3, $\lambda_{2,1}(K_2 \times K_2 \times K_2) = nm+2 = 6$. And $\lambda_{2,1}(K_3 \times K_2 \times K_2) = \lambda_{2,1}(C_3 \times C_4) = 8$ by [11].

The next section determines the $\sigma_{2,1}$-number of $K_n \times K_m \times K_l$ for any three positive integers n,m,l. We shall always suppose that n, m and l are positive integers with $n \ge m \ge l \ge 2$.

2. $\sigma_{2,1}(K_n \times K_m \times K_l)$

For two positive integers a and b with $a < b$, denote by $[a,b]$ the set of integers $a, a+1, \ldots, b$. A set of integers is called k-separated if any two distinct elements of the set differ by at least k. Given a graph $G(V,E)$, a subset S of V is called 2-independent if any two vertices in it are at distance at least 3. The 2-independence number of G is the number of vertices in a maximum 2-independent set of G.

We shall view the vertices of the graph $K_n \times K_m \times K_l$ as points in Euclidean three-space with coordinate (a,b,c), where a,b,c are nonnegative integers and $0 \le a \le n-1$, $0 \le b \le m-1$, $0 \le c \le l-1$. For $v=(a,b,c) \in V(K_n \times K_m \times K_l)$, we say that v is a vertex in the a^{th} row, b^{th} column and the c^{th} level of $K_n \times K_m \times K_l$. For fixed h, $0 \le h \le m-1$, we shall refer to the vertices on the 0^{th} level in the set $D_h = \{(a,b,0) | (b-a \bmod m) \bmod m = h\}$ as vertices along the h^{th} diagonal.

It is not difficult to see that two vertices are at distance k if their coordinates are different in exactly k components. In other words, two vertices on a line parallel to some coordinate axis are adjacent; two vertices on a plane parallel to some coordinate plane but not on any line parallel to some coordinate axis are at distance 2; and any two vertices not on any plane parallel to some coordinate plane are at distance 3. The diameter of $K_n \times K_m \times K_l$ is 3. The 2-independence number of $K_n \times K_m \times K_l$ is l. Thus each label can be used at most l times by any $L(2,1)$-labeling of $K_n \times K_m \times K_l$.

We first deal with the case that $n \geq 4$.

Theorem 2.1 *Let n, m and l be positive integers with $n \geq m \geq l \geq 2$. If $n \geq 4$ and $m \geq 3$ or $n \geq 5$ and $m = 2$, then*

$$\sigma_{2,1}(K_n \times K_m \times K_l) = \lambda_{2,1}(K_n \times K_m \times K_l) + 1 = nm.$$

Proof We split the proof into the following four cases.

Case 1 $n \geq m \geq 6$ or $n > m = 5$.

In the proof of Theorem 2.1 in [17], the matrix $X = (x_{ij})_{n\times m}$ was defined as:

$$X = \begin{pmatrix} nm-1 & 1 & 3 & 6 & \cdots & \cdots & \cdots \\ 2 & 4 & 7 & \cdots & \cdots & \cdots & \cdots \\ 5 & 8 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 9 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & nm-3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & nm-2 & 0 \end{pmatrix}_{n\times m} \equiv (x_{ij})_{n\times m}. \qquad (2.1)$$

Using this matrix, the authors defined a $\lambda_{2,1}$-labeling f with span $mn-1$ as:

$f((a,b,0)) = x_{(a+1)(b+1)}$, for $0 \leq a \leq n-1$, $0 \leq b \leq m-1$;

$f((a,b,c)) = f(((a+c) \bmod n, (b+c) \bmod m, 0))$, for $0 \leq a \leq n-1$, $0 \leq b \leq m-1$, $0 \leq c \leq l-1$.

We shall obtain the $\sigma_{2,1}$-circular labeling of $K_n \times K_m \times K_l$ by modifying the definition of the matrix X and using the same way to define the labeling f.

If $n \geq m \geq 6$, then let $x_{nm} = x_{25}$ and $x_{25} = 0$, and if $n > m = 5$, then let $x_{nm} = x_{2m}$ and $x_{2m} = 0$. Similar to Case 1 in the proof of Theorem 2.1 in [17], one can prove that f is an $L(2,1)$-labeling of $K_n \times K_m \times K_l$ with span $nm-1$. Furthermore, it is also easy to check that the vertices labeled by 0 and those labeled by $nm-1$ are nonadjacent. It follows that this $\lambda_{2,1}$-labeling of $K_n \times K_m \times K_l$ is also an mn-$(2,1)$-circular labeling of $K_n \times K_m \times K_l$. By Theorem 1.6, we have $\sigma_{2,1}(K_n \times K_m \times K_l) = nm$.

Case 2 $n > m = 4$.

In this case, we define the matrix X as:

$$\begin{pmatrix} 0 & 3n & 2n & n \\ 3n+1 & 2n+1 & n+1 & 1 \\ 2n+2 & n+2 & 2 & 3n+2 \\ n+3 & 3 & 3n+3 & 2n+3 \\ \cdots & \cdots & \cdots & \cdots \\ n-4 & 4n-4 & 3n-4 & 2n-4 \\ 4n-3 & 3n-3 & 2n-3 & n-3 \\ 3n-2 & 2n-2 & n-2 & 4n-2 \\ 2n-1 & n-1 & 4n-1 & 3n-1 \end{pmatrix}$$

a. $n = 0 \mod 4$

$$\begin{pmatrix} 0 & 3n & 2n & n \\ 3n+1 & 2n+1 & n+1 & 1 \\ 2n+2 & n+2 & 2 & 3n+2 \\ n+3 & 3 & 3n+3 & 2n+3 \\ \cdots & \cdots & \cdots & \cdots \\ 4n-4 & 3n-4 & 2n-4 & n-4 \\ 3n-3 & 2n-3 & n-3 & 4n-3 \\ 2n-2 & n-2 & 4n-2 & 3n-2 \\ n-1 & 4n-1 & 3n-1 & 2n-1 \end{pmatrix}$$

b. $n = 1 \mod 4$

$$\begin{pmatrix} 0 & n & 2n & 3n \\ n+1 & 2n+1 & 3n+1 & 1 \\ 2n+2 & 3n+2 & 2 & n+2 \\ 3n+3 & 3 & n+3 & 2n+3 \\ \cdots & \cdots & \cdots & \cdots \\ 3n-4 & 4n-4 & n-4 & 2n-4 \\ 4n-3 & n-3 & 2n-3 & 3n-3 \\ n-2 & 2n-2 & 3n-2 & 4n-2 \\ 2n-1 & 3n-1 & 4n-1 & n-1 \end{pmatrix}$$

c. $n = 2 \mod 4$

$$\begin{pmatrix} 0 & n & 2n & 3n \\ n+1 & 2n+1 & 3n+1 & 1 \\ 2n+2 & 3n+2 & 2 & n+2 \\ 3n+3 & 3 & n+3 & 2n+3 \\ \cdots & \cdots & \cdots & \cdots \\ 4n-4 & n-4 & 2n-4 & 3n-4 \\ n-3 & 2n-3 & 3n-3 & 4n-3 \\ 2n-2 & 3n-2 & 4n-2 & n-2 \\ 3n-1 & 4n-1 & n-1 & 2n-1 \end{pmatrix}$$

d. $n = 3 \mod 4$

Similar to Case 1, using these matrices, we can get mn-$(2,1)$-circular labelings of $K_n \times K_m \times K_l$.

Case 3 $n > m = 3$.

In this case, we define the matrix X as:

$$\begin{pmatrix} 0 & n+2 & 2n+1 \\ n & 2n+2 & 1 \\ 2n & 2 & n+1 \\ \cdots & \cdots & \cdots \\ n-3 & 2n-1 & 3n-1 \\ 2n-3 & 3n-2 & n-2 \\ 3n-3 & n-1 & 2n-2 \end{pmatrix}$$

a. $n = 0 \mod 3$

$$\begin{pmatrix} 0 & 2n & n \\ 2n+1 & n+1 & 1 \\ n+2 & 2 & 2n+2 \\ 3 & 2n+3 & n+3 \\ \cdots & \cdots & \cdots \\ 3n-3 & 2n-3 & n-3 \\ 2n-2 & n-2 & 3n-2 \\ n-1 & 3n-1 & 2n-1 \end{pmatrix}$$

b. $n = 1 \mod 3$

$$\begin{pmatrix} 0 & n & 2n \\ n+1 & 2n+1 & 1 \\ 2n+2 & 2 & n+2 \\ 3 & n+3 & 2n+3 \\ \cdots & \cdots & \cdots \\ 3n-3 & n-3 & 2n-3 \\ n-2 & 2n-2 & 3n-2 \\ 2n-1 & 3n-1 & n-1 \end{pmatrix}$$

c. $n = 2 \mod 3$

Similar to Case 1, using these matrices, we can get mn-$(2,1)$-circular labelings of $K_n \times K_m \times K_l$.

Case 4 $n = m = 5$ or $n = m = 4$.

We define the matrix X as:

$$\begin{pmatrix} 9 & 13 & 17 & 21 & 0 \\ 14 & 18 & 22 & 1 & 5 \\ 19 & 23 & 2 & 6 & 10 \\ 24 & 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 & 20 \end{pmatrix}$$

a. $n = m = 5$

$$\begin{pmatrix} 7 & 10 & 13 & 0 \\ 11 & 14 & 1 & 4 \\ 15 & 2 & 5 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

b. $n = m = 4$

Similar to Case 1, using these matrices, we can get mn-$(2,1)$-circular labelings of $K_n \times K_m \times K_l$.

Case 5 $n \geq 5$ and $m = 2$.

Clearly, we have $K_n \times K_2 \times K_2 \cong K_n \times C_4$. The $L(2,1)$-labeling of $K_n \times C_4$ with span $2n-1$ given by the following matrix[17] is also the $\sigma_{2,1}$-circular labeling of $K_4 \times K_2 \times K_2$. □

We now consider the case $n = 4$ and $m = 2$.

Theorem 2.2 $\sigma_{2,1}(K_4 \times K_2 \times K_2) = \lambda_{2,1}(K_4 \times K_2 \times K_2) + 2 = 10$.

Proof In [17], we know that $\lambda_{2,1}(K_4 \times K_2 \times K_2) = 8$. Let $V_{jk} = \{(a,j,k) | 0 \leq a \leq 3\}$ for $j = 0,1$ and $k = 0,1$. Suppose $\sigma_{2,1}(K_4 \times K_2 \times K_2) = 9$. Let g be a 9-circular labelings of $K_4 \times K_2 \times K_2$. Define X_{i1} and X_{i1} as follows:

$X_{01} = \{1,3,5,7\}$ and $X_{01} = \{2,4,6,8\}$; $X_{11} = \{0,3,5,7\}$ and $X_{11} = X_{01}$;

$X_{21} = X_{11}$, $X_{21} = \{1,4,6,8\}$; $X_{31} = \{0,2,5,7\}$, $X_{31} = X_{21}$;

$X_{41} = X_{31}$, $X_{41} = \{1,3,6,8\}$; $X_{51} = \{0,2,4,7\}$, $X_{51} = X_{41}$;

$X_{61} = X_{51}$, $X_{61} = \{1,3,6,8\}$; $X_{71} = \{0,2,4,6\}$, $X_{71} = X_{61}$;

$X_{81} = X_{71}$, $X_{81} = X_{11}$.

Then for $k = 0,1$, $f(V_{0k}) \cup f(V_{1k}) = X_{i1} \cup X(i2)$ for some $0 \leq i \leq 8$. Clearly, there must exist four consecutive labels i, $i+1$, $i+2$ and $i+3$ in $f(V_{00}) \cup f(V_{10})$ and $f(V_{01}) \cup f(V_{11})$. Then $i+4$ and $i-1$ must not be used in the two levels of $K_4 \times K_2 \times K_2$, where the "+" and "-" are taken modulo 9, a contradiction.

So $\sigma_{2,1}(K_4 \times K_2 \times K_2) = 10$. □

We now turn to the case $n \leq 3$.

Theorem 2.3 $\sigma_{2,1}(K_2 \times K_2 \times K_2) = \lambda_{2,1}(K_2 \times K_2 \times K_2) + 2 = 8$.

Proof Let f be any k-$(2,1)$-circular labeling of K_2^3. For $i \in [0, k-1]$, it is easy to see that i can be assigned to at most two vertices of K_2^3. Furthermore, if i is assigned to two vertices, then $i-1$ and $i+1$ (where "$-$" and "+" are taken modulo k) cannot be assigned to any vertices of K_2^3. For $i \in [0, k-1]$, let $A_i = \{v | f(v) = i, \text{or}\, i+1 \,\text{and}\, v \in V(K_2^3)\}$. It follows from the above discussion that $|A_i| \leq 2$ for each $i \in [0, k-1]$. Therefore $\sum_{i=0}^{k-1} |A_i| \leq 2k$. On the other hand, since K_2^3 has 8 vertices, we clearly have $\sum_{i=0}^{k-1} |A_i| = 16$. This implies $k \geq 8$. By Theorem 1.6, $\sigma_{2,1}(K_2 \times K_2 \times K_2) = \lambda_{2,1}(K_2 \times K_2 \times K_2) + 2 = 8$. □

Next we consider the case $n = 3$.

Theorem 2.4 *For $n = 3$ and $m = l = 2$, we have $\sigma_{2,1}(K_n \times K_m \times K_l) = 9$.*

Proof Note that $K_3 \times K_2 \times K_2 \cong C_3 \times C_4$. So $\sigma_{2,1}(K_3 \times K_2 \times K_2) = \sigma_{2,1}(C_3 \times C_4)$. In [11], a $\lambda_{2,1}$- labeling of $C_3 \times C_4$ is defined by the matrix Y as follows:

$$Y = \begin{pmatrix} 6 & 4 & 0 & 2 \\ 3 & 1 & 6 & 8 \\ 0 & 7 & 3 & 5 \end{pmatrix}. \tag{2.2}$$

It is also a 9-circular labeling of $C_3 \times C_4$. Thus we have that $\sigma_{2,1}(C_3 \times C_4) = 9$. Then $\sigma_{2,1}(K_3 \times K_2 \times K_2) = 9$. □

Finally we assume that $n = m = 3$.

Theorem 2.5

$$\sigma_{2,1}(K_3 \times K_3 \times K_l) = \lambda_{2,1}(K_3 \times K_3 \times K_l) + 2 = \begin{cases} 11, & \text{if } l = 2, \\ 12, & \text{if } l = 3. \end{cases}$$

Proof By Theorems 1.6 and 2.5, we have $\sigma_{2,1}(K_3 \times K_3 \times K_2) \leq 11$ and $\sigma_{2,1}(K_3 \times K_3 \times K_3) \leq 12$. To prove the theorem, it suffices to show that there is no k-$(2,1)$-circular labeling of $K_3 \times K_3 \times K_2$ with $k < 11$ and there is no k-$(2,1)$-circular labeling of $K_3 \times K_3 \times K_3$ with $k < 12$.

Let f be a k-$(2,1)$-circular labeling of $K_3 \times K_3 \times K_2$. As in the proof of Theorem 2.3 in [17], we can make the following observation.

Observation A For any integer $i \in [0, k-1]$, if each of the three consecutive labels $i-1, i$ and $i+1$ is assigned to exactly two vertices of $K_3 \times K_3 \times K_2$, then the three vertices in the same level receiving the labels $i-1, i$, and $i+1$ respectively must lie in different rows and different columns, i.e., the three vertices in the same level receiving labels $i-1, i$, and $i+1$ are along some diagonal. (Note that vertices in each level can be partitioned into three disjoint diagonals.)

Then each label is used at most twice by f. From the above observation, any four consecutive labels are assigned to at most 7 vertices. For $i \in [0, k-1]$, let $A_i = \{v | f(v) \in \{i, i+1, i+2, i+3\} \text{ and } v \in V(K_2^3)\}$ (where "+" is taken modulo k). Then $|A_i| \leq 7$ for each $i \in [0, k-1]$ and so $\sum_{i=0}^{k-1} |A_i| \leq 7k$. As $K_3 \times K_3 \times K_2$ has 18 vertices, we must have $\sum_{i=0}^{k-1} |A_i| = 4 \times 18 = 72$. It follows that $k \geq 11$.

We now deal with the graph K_3^3. Let f be a k-$(2,1)$-circular labeling of K_3^3. For $i \in [0, k-1]$, let m_i be the number of vertices v of K_3^3 with $f(v) = i$. Clearly $0 \leq m_i \leq 3$ for $i \in [0, k-1]$ and $\sum_{i=0}^{k-1} m_i = 27$. By Observation A, it is not difficult to make the following three observations.

Observation B For any integer $i \in [0, k-1]$, if $m_i = m_{i+1} = m_{i+2} = 3$, then $m_{i-1} = m_{i+3} = 0$.

Observation C For any integer $i \in [0, k-1]$, if $m_i = 2$ and $m_{i+1} = m_{i+2} = 3$, then $m_{i+3} \leq 1$.

Observation D For any integer $i \in [0, k-1]$, if $m_i = m_{i+2} = 3$ and $m_{i+1} = 2$, then $m_{i+3} \leq 1$.

It follows from Observations B, C and D that $\sum_{j=i}^{i+3} m_j \leq 10$ for any $i \in [0, k-1]$. Furthermore, $\sum_{j=i}^{i+3} m_j = 10$ if and only if $(m_i, m_{i+1}, m_{i+2}, m_{i+3})$ is one of the following forms: $(3,2,2,3)$, $(3,3,2,2)$, $(2,2,3,3)$, $(3,3,1,3)$, $(3,1,3,3)$.

Next we show that if $k \le 11$, then $\sum_{i=0}^{k-1} m_i < 27$ and thus get a contradiction.

If $m_i \ge 2$ for all $i \in [0, k-1]$, then, by Observations C and D, it is easy to see that there are at most three integers i with $m_i = 3$ and so $\sum_{i=0}^{k-1} m_i < 27$. Now suppose w.l.o.g. that $m_0 \le 1$. If $m_0 + m_1 + m_2 \le 6$, then since $\sum_{j=i}^{i+3} m_j \le 10$ for $i = 3, 7$, $\sum_{i=0}^{k-1} m_i < 27$. Thus we assume $m_0 + m_1 + m_2 \ge 7$. Then we must have $m_0 = 1$ and $m_1 = m_2 = 3$. If $\sum_{j=i}^{i+3} m_j \le 9$ for $i = 3$ or 7, then $\sum_{i=0}^{k-1} m_i < 27$. Thus we assume that $\sum_{j=i}^{i+3} m_j = 10$ for $i = 3, 7$. This implies that (m_3, m_4, m_5, m_6) and (m_7, m_8, m_9, m_{10}) must be of the form $(2,2,3,3)$. But then $(m_4, m_5, m_6, m_7) = (2,3,3,2)$. This is a contradiction to Observation C.

References

[1] CHANG G J, KUO D. *The $L(2,1)$-labeling problem on graphs* [J]. SIAM J. Discrete Math., 1996, **9**(2): 309–316.

[2] GEORGES J P, MAURO D W. *Generalized vertex labelings with a condition at distance two* [J]. Congr. Numer., 1995, **109**: 141–159.

[3] GEORGES J P, MAURO D W. *Some results on λ_k^j-numbers of the products of complete graphs* [J]. Congr. Numer., 1999, **140**: 141–160.

[4] GEORGES J P, MAURO D W. *Labeling trees with a condition at distance two* [J]. Discrete Math., 2003, **269**(1-3): 127–148.

[5] GEORGES J P, MAURO D W, STEIN M I. *Labeling products of complete graphs with a condition at distance two* [J]. SIAM J. Discrete Math., 2000, **14**: 28-35.

[6] GEORGES J P, MAURO D W, WHITTLESEY M A. *Relating path coverings to vertex labellings with a condition at distance two* [J]. Discrete Math., 1994, **135**(1-3): 103–111.

[7] GRIGGS J R, YEH R K. *Labelling graphs with a condition at distance* 2 [J]. SIAM J. Discrete Math., 1992, **5**(4): 586–595.

[8] HEUVEL J, LEESE R A, SHEPHERD M A. *Graph labeling and radio channel assignment* [J]. J. Graph Theory, 1998, **29**(4): 263–283.

[9] JHA P K, NARAYANAN A, SOOD P. et al. *On $L(2,1)$-labeling of the Cartesian product of a cycle and a path* [J]. Ars Combin., 2000, **55**: 81–89.

[10] KLAVŽAR S, VESEL A. *Computing graph invariants on rotagraphs using dynamic algorithm approach: the case of $(2,1)$-colorings and independence numbers* [J]. Discrete Appl. Math., 2003, **129**(2-3): 449–460.

[11] KUO D, YAN J. *On L(2,1)-labelings of Cartesian products of paths and cycles* [J]. Discrete Math., 2004, **283**: 137–144.

[12] SAKAI D. *Labelling chordal graphs: distance two condition* [J]. SIAM J. Discrete Math., 1994, **7**: 133–140.

[13] SCHWARZAND C, TROXELL D S. *L(2,1)-Labelings of Products of Two Cycles* [C]. DIMACS Technical Report 2003-33, 2003.

[14] WHITTLESEY M A, GEORGES J P, MAURO D W. *On the λ-number of Q_n and related graphs* [J]. SIAM J. Discrete Math., 1995, **8**(4): 499–506.

[15] LIU D D F. *Hamiltoncity and circular distance two labelings* [J]. Discrete Math., 2001, **232**: 163–169.

[16] LAM P C B, LIN Wensong, WU Jianzhuan. *$L(j,k)$-and circular $L(j,k)$-labellings for the products of complete graphs* [J]. J. Comb. Optim., 2007, **14**(2-3): 219–227.

[17] DAMEI L, LIN Wensong, SONG Zengmin. *Distance two labelings of Cartesian products of complete graphs* [J]. manuscript, 2006.