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# Stable Rings for Morita Contexts of Generalized Power Series Rings

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**Abstract** In this paper, we show that if rings $A$ and $B$ are $(s,2)$-rings, then so is the ring of a Morita Context $([[A^{S,\leq}]],[[B^{S,\leq}]],[[M^{S,\leq}]],[[N^{S,\leq}]],\psi^S,\phi^S)$ of generalized power series. Also we get analogous results for unit 1-stable ranges, $GM$-rings and rings which have stable range one. These give new classes of rings satisfying such stable range conditions.

**Keywords** Morita Context; unit 1-stable range; generalized power series.

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## 1. Introduction

Let $R$ be an associative ring with identity and $(S,\leq)$ a strictly ordered monoid. Let $[[R^{S,\leq}]]$ be the set of all maps $f:S\longrightarrow R$ such that $\operatorname{supp}(f)=\{s\in S|f(s)\neq 0\}$ is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s\in S$ and $f,g\in[[R^{S,\leq}]]$, let $X_s(f,g)=\{(u,v)\in S\times S|s=u+v,f(u)\neq 0,g(v)\neq 0\}$, it follows from [1, Section 4.1] that $X_s(f,g)$ is finite. This fact allows to define the multiplication:

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)}f(u)g(v).$$

With such multipliation and the preceeding pointwise addition, $[[R^{S,\leq}]]$ turns out to be a ring with unit element $e^*$ given by $e^*(0)=1,e^*(s)=0$ for all $0\neq s\in S$. such ring is called a ring of generalized power series.

The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in $R$ and exponents in $S$. For any a $\in R$, $C_a\in[[R^{S,\leq}]]$ is given by $C_a(0)=a,C_a(s)=0$ for all $0\neq s\in S$. Ordered monoids $(S,\leq)$ is said to satisfy condition $(S0)$ in case $s\geq 0$ for all $s\in S$. Henceforth, unless otherwise mentioned, in this paper, $(S,\leq)$ will always denote a strictly ordered monoid which satisfies condition (S0).

Let $M$ be an $R$-module. $[[M^{S,\leq}]]$ denotes the set of all maps $\phi:S\longrightarrow M$ such that $\operatorname{supp}(\phi)=\{s\in S|\phi(s)\neq 0\}$ is artinian and narrow. From [2], it is immediate that $[[M^{S,\leq}]]$ is

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an $[[R^{S,\leq}]]$-module. For any $f \in [[R^{S,\leq}]], \phi \in [[M^{S,\leq}]]$ and $s \in S$, the scalar multiplication is defined as follows:

$$(f\phi)(s) = \sum_{(u,v)\in X_s(f,\phi)} f(u)\phi(v).$$

A ring $R$ is said to be a $(s,2)$-ring in case every element of $R$ is the sum of two units. We say that $R$ satisfies unit 1-stable range provided that $aR + bR = R$ implies that $a + bu \in U(R)$ for a $u \in U(R)$. A ring $R$ is said to be a $GM$-ring provided that for any $x, y \in R$, there exist $e^2 = e, f^2 = f \in R$ and $u \in U(R)$ such that $x - eu, y - fu^{-1} \in U(R)$.

Recall that a Morita Context denoted by $(A, B, N, M, \psi, \phi)$ consists of two rings $A, B$, two bimodules ${}_AN_B, {}_BM_A$ and a pair bimodule homomorphisms $\psi : N \bigotimes_B M \longrightarrow A$ and $\phi : M \bigotimes_A N \longrightarrow B$ which satisfy the following associativity: $\psi(v, w)v' = v\phi(w, v')$ and $\phi(w, v)w' = w\psi(v, w')$. These conditions will insure that the set $T$ of generalized matrices

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix}; a \in A, b \in B, m \in M, n \in N$$

will form a ring, called the ring of the Morita Context.

In this paper, we show that if $(A, B, N, M, \psi, \phi)$ is a Morita context, then $([[A^{S,\leq}]], [[B^{S,\leq}]], [[M^{S,\leq}]], [[N^{S,\leq}]], \psi^S, \phi^S)$, where $\psi^S : [[N^{S,\leq}]] \bigotimes_{[[B^{S,\leq}]]} [[M^{S,\leq}]] \longrightarrow [[A^{S,\leq}]]$, and $\phi^S : [[M^{S,\leq}]] \bigotimes_{[[A^{S,\leq}]]} [[N^{S,\leq}]] \longrightarrow [[B^{S,\leq}]]$ which satisfy the following associativity: $\psi^S(n, m)n' = n\phi^S(m, n')$, $\phi^S(m, n)m' = m\psi^S(n, m')$ for all $n, n' \in [[N^{S,\leq}]], m, m' \in [[M^{S,\leq}]]$ is a Morita context. The set $T^S$ of generalized matrices

$$\begin{pmatrix} f & n \\ m & g \end{pmatrix}; a \in [[A^{S,\leq}]], b \in [[B^{S,\leq}]], m \in [[M^{S,\leq}]], n \in [[N^{S,\leq}]]$$

will form a ring, called the ring of the Morita Context of generalized power series. Furthermore, we show that if ring $A$ and $B$ are $(s,2)$-rings, then so is the ring $T^S = \begin{pmatrix} [[A^{S,\leq}]] & [[N^{S,\leq}]] \\ [[M^{S,\leq}]] & [[B^{S,\leq}]] \end{pmatrix}$ of Morita Context $([[A^{S,\leq}]], [[B^{S,\leq}]], [[M^{S,\leq}]], [[N^{S,\leq}]], \psi^S, \phi^S)$. Also we get analogous results for unit 1-stable regular rings, rings which have stable range one and $GM$-rings over Morita Contexts. These give new classes of rings statisfying such stable range conditions.

Throughtout, all rings are associative with identity and all modules are unitary. $U(R)$ denotes the group of units of $R$, $T$ always denotes the ring $\begin{pmatrix} A & N \\ M & B \end{pmatrix}$ of a Morita Context $(A, B, N, M, \psi, \phi)$, and $T^S$ the ring $\begin{pmatrix} [[A^{S,\leq}]] & [[N^{S,\leq}]] \\ [[M^{S,\leq}]] & [[B^{S,\leq}]] \end{pmatrix}$ of a Morita Context

$$([[A^{S,\leq}]], [[B^{S,\leq}]], [[M^{S,\leq}]], [[N^{S,\leq}]], \psi^S, \phi^S).$$

## 2. Main results

**Theorem 2.1** *Let $(A, B, N, M, \psi, \phi)$ be a Morita Context. Then there exist a pair of bimodule* *homomorphisms* $\psi^S : [[N^{S,\leq}]] \bigotimes_{[[B^{S,\leq}]]} [[M^{S,\leq}]] \longrightarrow [[A^{S,\leq}]]$ *and* $\phi^S : [[M^{S,\leq}]] \bigotimes_{[[A^{S,\leq}]]} [[N^{S,\leq}]] \longrightarrow [[B^{S,\leq}]]$ *such that* $([[A^{S,\leq}]], [[B^{S,\leq}]], [[M^{S,\leq}]], [[N^{S,\leq}]], \psi^S, \phi^S)$ *is a Morita Context.*

**Proof** Since $N$ is a left $A$-right $B$-bimodule, and $M$ is a left $B$-right $A$-bimodule, by [2], we have $[[N^{S,\leq}]]$ is a left $[[A^{S,\leq}]]$-right $[[B^{S,\leq}]]$-bimodule and $[[M^{S,\leq}]]$ is a left $[[B^{S,\leq}]]$-right $[[A^{S,\leq}]]$-bimodule.

Consider the following diagram:

$$\begin{array}{ccc} [[N^{S,\leq}]] \times [[M^{S,\leq}]] & \xrightarrow{\pi} & [[N^{S,\leq}]] \bigotimes_{[[B^{S,\leq}]]} [[M^{S,\leq}]] \\ \Big\downarrow f & \swarrow \psi^S & \\ [[A^{S,\leq}]] & & \end{array}$$

Let $n \in [[N^{S,\leq}]]$ and $m \in [[M^{S,\leq}]]$. Define a map $\alpha_{[n,m]} : S \longrightarrow A$.

$$\alpha_{[n,m]}(s) = \sum_{(u,v)\in X_s(n,m)} \psi(n(u), m(v))$$

for any $s \in S$. It is clear that $\mathrm{supp}(\alpha_{[n,m]}) \subseteq \mathrm{supp}(n) + \mathrm{supp}(m)$, thus $\alpha_{[n,m]} \in [[A^{S,\leq}]]$.

Define a map $f : [[N^{S,\leq}]] \times [[M^{S,\leq}]] \longrightarrow [[A^{S,\leq}]]$, where $f((n,m)) = \alpha_{[n,m]}$ for any $(n,m) \in [[N^{S,\leq}]] \times [[M^{S,\leq}]]$. Let $n_1, n_2 \in [[N^{S,\leq}]]$, $m \in [[M^{S,\leq}]]$. By the preceding discussions, there exist $\alpha_{[n_1,m]}, \alpha_{[n_2,m]}, \alpha_{[n_1+n_2,m]} \in [[A^{S,\leq}]]$. For all $s \in S$,

$$\begin{aligned} \alpha_{[n_1+n_2,m]}(s) &= \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi((n_1+n_2)(u), m(v)) \\ &= \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi(n_1(u), m(v)) + \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi(n_2(u), m(v)). \end{aligned}$$

If $(u',v') \in X_s(n_1,m)$, but $(u',v') \bar{\in} X_s(n_1+n_2,m)$, then $(n_1+n_2)(u') = 0$. So $n_2(u') \neq 0$, thus $(u',v') \in X_s(n_2,m)$ and $\psi(n_1(u'), m(v')) + \psi(n_2(u'), m(v')) = \psi(n_1(u') + n_2(u'), m(v')) = 0$. Likewise, if $(u',v') \in X_s(n_2,m)$, but $(u',v') \bar{\in} X_s(n_1+n_2,m)$, we also have $(u',v') \in X_s(n_1,m)$ and $\psi(n_1(u'), m(v')) + \psi(n_2(u'), m(v')) = \psi(n_1(u') + n_2(u'), m(v')) = 0$. So

$$\begin{aligned} \alpha_{[n_1+n_2,m]}(s) &= \sum_{(u,\,v)\in X_s(n_1+n_2,m)} \psi(n_1(u), m(v)) + \sum_{(u,\,v)\in X_s(n_1+n_2,m)} \psi(n_2(u), m(v)) \\ &= \sum_{(u,v)\in X_s(n_1,\,m)} \psi(n_1(u), m(v)) + \sum_{(u,v)\in X_s(n_2,m)} \psi(n_2(u), m(v)) \\ &= \alpha_{[n_1,m]}(s) + \alpha_{[n_2,m]}(s) = (\alpha_{[n_1,m]} + \alpha_{[n_2,m]})(s). \end{aligned}$$

Thus $\alpha_{[n_1+n_2,m]} = \alpha_{[n_1,m]} + \alpha_{[n_2,m]}$, hence $f((n_1+n_2,m)) = f((n_1,m)) + f((n_2,m))$. By the same manner, we see that $f((n, m_1+m_2)) = f((n,m_1)) + f((n,m_2))$ for all $n \in [[N^{S,\leq}]], m_1, m_2 \in [[M^{S,\leq}]]$.

For any $n \in [[N^{S,\leq}]], \tau \in [[B^{S,\leq}]], m \in [[M^{S,\leq}]]$ and any $s \in S$, we have

$$\begin{aligned} f((n\tau, m))(s) =& \alpha_{[n\tau, m]}(s) \\ =& \sum_{(u',u)\in X_s(n\tau,m)} \psi(n\tau(u'), m(u)) \\ =& \sum_{(u',u)\in X_s(n\tau,m)} \psi(\sum_{(v,w)\in X_{u'}(n,\tau)} (n(v)\tau(w), m(u))) \\ =& \sum_{(u',u)\in X_s(n\tau,m)} \sum_{(v,w)\in X_{u'(n,\tau)}} \psi(n(v)\tau(w), m(u)) \\ =& \sum_{(u',u)\in X_s(n\tau,m)} \sum_{(v,w)\in X_{u'}(n,\tau)} \psi(n(v)\tau(w), m(u)) + \\ & \sum_{(v,w,u)\in X} \psi(n(v)\tau(w), m(u)) \\ =& \sum_{(v,w,u)\in X_s(n,\tau,m)} \psi(n(v)\tau(w), m(u)) \\ =& \sum_{(v,w,u)\in X_s(n,\tau,m)} \psi(n(v), \tau(w)m(u)) \\ =& f((n, \tau m))(s), \end{aligned}$$

where $X = \{(v, w, u) \in X_s(n, \tau, m) | n\tau(v + w) = 0\}$. Thus $f(n\tau, m) = f(n, \tau m)$ and hence $f$ is a bilinear balanced morphism. Then there exists a homomorphism $\psi^S : [[N^{S,\leq}]] \bigotimes_{[[B^{S,\leq}]]} [[M^{S,\leq}]] \longrightarrow [[A^{S,\leq}]]$ such that the preceding diagram commutes.

Next, we check that $\psi^S$ is a bimodule homomorphism. For any $a \in [[A^{S,\leq}]], n \in [[N^{S,\leq}]], m \in [[M^{S,\leq}]]$ and any $s \in S$,

$$\begin{aligned} \psi^S(an,\, m)(s) =& \alpha_{[an,\, m]}(s) \\ =& \sum_{(u',u)\in X_s(an,m)} \psi(an(u'), m(u)) \\ =& \sum_{(u',u)\in X_s(an,m)} \psi(\sum_{(v,w)\in X_{u'}(a,\, n)} (a(v)n(w), m(u))) \\ =& \sum_{(v,w,u)\in X_s(a,n,m)} \psi(a(v)n(w), m(u)) \\ =& \sum_{(v,w,u)\in X_s(a,n,m)} a(v)\psi(n(w), m(u)) \\ =& a\psi^S(n, m)(s). \end{aligned}$$

Thus $\psi^S(an, m) = a\psi^S(n, m)$. This implies that $\psi^S$ is a left $[[A^{S,\leq}]]$-module homomorphism. Analogously, it is easy to verify that $\psi^S$ is a right $[[B^{S,\leq}]]$-module homomorphism. Thus $\psi^S$ is a bimodule homomorphism. Likewise, we claim that there exists a bimodule homomorphism: $\phi^S : [[M^{S,\leq}]] \bigotimes_{[[A^{S,\leq}]]} [[N^{S,\leq}]] \longrightarrow [[B^{S,\leq}]]$. For any $n, n' \in [[N^{S,\leq}]], m \in [[M^{S,\leq}]]$ and any $s \in S$, we have

$$\psi^S(n, m)n'(s) = \alpha_{[n,m]}n'(s)$$

$$
\begin{aligned}
&= \sum_{(u',u)\in X_s(\alpha_{[n,m]},n')} \alpha_{[n,m]}(u')n'(u) \\
&= \sum_{(u',u)\in X_s(\alpha_{[n,m]},n')} \Big( \sum_{(v,w)\in X_{u'}(n,m)} \psi(n(v),m(w))\Big) n'(u) \\
&= \sum_{(u',u)\in X_s(\alpha_{[n,m]},n')} \sum_{(v,w)\in X_{u'}(n,m)} \psi(n(v),m(w))n'(u) + \\
&\quad \sum_{(v,w,u)\in X} \psi(n(v),m(w))n'(u) \\
&= \sum_{(v,w,u)\in X_s(n,m,n')} \psi(n(v),m(w))n'(u) \\
&= \sum_{(v,w,u)\in X_s(n,m,n')} n(v)\phi(m(w),n'(u)) \\
&= n\phi^S(m,n')(s),
\end{aligned}
$$

where $X = \{(v,w,u) \in X_s(n,mn') | \alpha_{[n,m]}(v+w) = 0\}$. Thus $\psi^S(n,m)n' = n\phi^S(m,n')$. Analogously, $\phi^S(m,n)m' = m\psi^S(n,m')$ for $m, m' \in [[M^{S,\leq}]]$ and $n \in [[N^{S,\leq}]]$. Thus

$$([[A^{S,\leq}]], [[B^{S,\leq}]], [[M^{S,\leq}]], [[N^{S,\leq}]], \psi^S, \phi^S)$$

is a Morita Context.

**Theorem 2.2** *Let* $T^S = \begin{pmatrix} [[A^{S,\leq}]] & [[N^{S,\leq}]] \\ [[M^{S,\leq}]] & [[B^{S,\leq}]] \end{pmatrix}$ *denote a ring of the Morita Context*

$$([[A^{S,\leq}]],\ [[B^{S,\leq}]],\ [[M^{S,\leq}]],\ [[N^{S,\leq}]],\ \psi^S,\ \phi^S).$$

*Then we have* $\begin{pmatrix} [[A^{S,\leq}]] & [[N^{S,\leq}]] \\ [[M^{S,\leq}]] & [[B^{S,\leq}]] \end{pmatrix} \cong \left[\left[\begin{pmatrix} A & N \\ M & B \end{pmatrix}^{S,\leq}\right]\right].$

**Proof** As in the proof of [1, Proposition 4.3], we complete the proof.

**Lemma 2.3**[3] *Let* $(S, \leq)$ *be a strictly ordered monoid which satisfies condition* $(S0)$*. Then* $f \in U([[A^{S,\leq}]])$ *if and only if* $f(0) \in U(R)$*.*

**Lemma 2.4**[5] *Let* $R$ *be a reduced commutative ring,* $(S, \leq)$ *a cancellative torsion-free monoid. Then* $\phi^2 = \phi \in [[R^{S,\leq}]]$ *if and only if there exists an* $e^2 = e \in R$ *such that* $\phi = C_e$*.*

**Theorem 2.5** *If* $A$ *and* $B$ *are* $(s, 2)$*-rings, then* $T^S$ *is a* $(s, 2)$*-ring.*

**Proof** Since $A$ is a $(s, 2)$-ring, there exist $u, v \in U(R)$ such that $a = u + v$ for any $a \in A$. Thus for all $f \in [[A^{S,\leq}]]$, $s \in S$, we have $f(s) = u_s + v_s$ where $u_s, v_s \in U(R)$. Let $f_1 : S \longrightarrow A$ be given by $f_1(s) = u_s$, and $f_2 : S \longrightarrow A$ be given by $f_2(s) = v_s$. Clearly, $f = f_1 + f_2$. Since $f(0) = u_0 + v_0$, where $u_0, v_0 \in U(R)$. By Lemma 2.3, we have $f_1, f_2 \in U([[A^{S,\leq}]])$. This implies that $[[A^{S,\leq}]] = U([[A^{S,\leq}]]) + U([[A^{S,\leq}]])$. Hence $[[A^{S,\leq}]]$ is a (s, 2)- ring. Likewise, we claim that $[[B^{S,\leq}]]$ is also a $(s, 2)$-ring. Thus by [3,Theorem1], $T^S = \begin{pmatrix} [[A^{S,\leq}]] & [[N^{S,\leq}]] \\ [[M^{S,\leq}]] & [[B^{S,\leq}]] \end{pmatrix}$ is a

$(s, 2)$-ring.

**Lemma 2.6** *A ring $R$ satisfies unit1-stable range if and only if $[[R^{S,\leq}]]$ satisfies unitl-stable range.*

**Proof** Assume that $R$ satisfies unit1-stable range. Let $f[[R^{S,\leq}]] + g[[R^{S,\leq}]] = [[R^{S,\leq}]]$ where $f, g \in [[R^{S,\leq}]]$, there exist $\tau, \omega \in [[R^{S,\leq}]]$ such that $f\tau + g\omega = e^*$. So $\sum_{(u,v)\in X_0(f,\tau)} f(u)\tau(v) + \sum_{(s,t)\in X_0(g,\omega)} g(s)\omega(t) = 1$. Since $(S, \leq)$ satisfies condition (S0), $u+v = 0$ if and only if $u = v = 0$, and $s + t = 0$ if and only if $s = t = 0$. So $f(0)\tau(0) + g(0)\omega(0) = 1$, thus $f(0)R + g(0)R = R$. Since $R$ satisfies unit1-stable rang, there exists a $u \in U(R)$ such that $f(0) + g(0)u \in U(R)$. Thus $f + gC_u \in U([[R^{S,\leq}]])$ where $C_u \in U([[R^{S,\leq}]])$. This implies that $[[R^{S,\leq}]]$ satisfies unit1-stable range.

Conversely, if $[[R^{S,\leq}]]$ statisfies unit1-stable range. Let $aR + bR = R$ for some $a, b \in R$, there exist $s, t \in R$ such that $as+bt = 1$. Then $C_aC_s+C_bC_t = e^*$, so $C_a[[R^{S,\leq}]]+C_b[[R^{S,\leq}]] = [[R^{S,\leq}]]$. Since $[[R^{S,\leq}]]$ satisfies unit1-stable range, there exists $f \in U([[R^{S,\leq}]])$ such that $C_a + C_bf \in U([[R^{S,\leq}]])$. Thus $(C_a + C_bf)(0) = a + bf(0) \in U(R)$. Therefore $R$ satisfies unit1-stable range.

**Theorem 2.7** *If $A$ and $B$ both satisfy unit 1-stable range. Then $T^S$ satisfies unit 1-stable range.*

**Proof** Suppose that $A$ and $B$ both satisfy unit 1-stable range. Then by [3, Theorem 5], $T$ satisfies unit 1-stable range. In view of Theorem 2.2 and Lemma 2.6, the result follows.

**Lemma 2.8** *Let $R$ be a reduced commutative ring, $(S, \leq)$ a cancellative torsion-free monoid. Then $R$ is a $GM$-ring if and only if $[[R^{S,\leq}]]$ is a $GM$-ring.*

**Proof** Suppose that $R$ is a $GM-$ ring. Let $f, g \in [[R^{S,\leq}]]$. Then $f(0), g(0) \in R$. There exist $e^2 = e, f^2 = f \in R$ and $u \in U(R)$ such that $f(0) - eu, g(0) - fu^{-1} \in U(R)$. As a result of $(f - C_eC_u)(0), (g - C_fC_{u^{-1}})(0) \in U(R)$ and $C_{u^{-1}} = C_u^{-1}$, $f - C_eC_u, g - C_fC_u^{-1} \in U([[R^{S,\leq}]])$ and $C_e^2 = C_e, C_f^2 = C_f, C_u \in U([[R^{S,\leq}]])$. Thus $[[R^{S,\leq}]]$ is a $GM$-ring.

Conversely, assume that $[[R^{S,\leq}]]$ is a $GM$-ring. Let $a, b \in R$. Then $C_a, C_b \in [[R^{S,\leq}]]$. Since $[[R^{S,\leq}]]$ is a $GM$-ring, there exist $e^2 = e, f^2 = f$ in $[[R^{S,\leq}]]$ and $\tau$ in $U([[R^{S,\leq}]])$ such that $C_a - e\tau, C_b - f\tau^{-1} \in U(([[R^{S,\leq}]])$. Since $f^2 = f$ and $e^2 = e$, we have $f(0)f(0) = f(0)$ and $e(0)e(0) = e(0)$. Thus, $(C_a-e\tau)(0) = a-e(0)\tau(0) \in U(R)$ and $(C_b-f\tau^{-1})(0) = b-f(0)\tau^{-1}(0) \in U(R)$. This implies that $R$ is a $GM$-ring.

**Theorem 2.9** *Let $A, B$ be reduced commutative rings, $(S, \leq)$ a cancellative torsion-free monoid. If $A, B$ are $GM$-rings, then $T^S$ is a GM-ring.*

**Proof** Since $A$ and $B$ are $GM$-ring, by [3, Theorem 8], $T$ is a $GM$-ring. Thus the result follows by Theorem 2.2 and Lemma 2.8.

**Lemma 2.10** *A ring $R$ has stable range one if and only if $[[R^{S,\leq}]]$ has stable range one.*

**Proof** Assume that $[[R^{S,\leq}]]$ has stable range one. Let $a, b \in R$ such that $as + bt = 1$ for some $s, t \in R$, then we have $C_aC_s + C_bC_t \in [[R^{S,\leq}]]$. Since $(C_aC_s + C_bC_t)(0) = as + bt = 1$, and $C_aC_s + C_bC_t)(s) = 0$ for each $s \neq 0$. Thus $C_aC_s + C_bC_t = e^*$. Since $[[R^{S,\leq}]]$ has stable range one, there exists $f \in [[R^{S,\leq}]]$ such that $C_a + C_bf \in U([[R^{S,\leq}]])$. Thus $(C_a + C_bf)(0) = a + bf(0) \in U(R)$, this imples that $R$ has stable range one.

Conversely, suppose that $R$ has stable range one. Let $f, \tau, g, \omega \in [[R^{S,\leq}]]$ such that $f\tau + g\omega = e^*$. Then $(f\tau + g\omega)(0) = f\tau(0) + g\omega(0) = f(0)\tau(0) + g(0)\omega(0) = 1$. Since $R$ has stable range one, there exists $x \in R$ such that $f(0) + g(0)x \in U(R)$. Thus $(f + gC_x)(0) \in U(R)$, this imples that $f + gC_x \in U([[R^{S,\leq}]])$. Therefore, $[[R^{S,\leq}]]$ has stable range one.

**Theorem 2.11** *$T^S$ has stable range one if and only if $A, B$ have stable range one.*

**Proof** Suppose $T^S$ has stable range one. Set $e = \begin{pmatrix} e^* & 0 \\ 0 & 0 \end{pmatrix}$. Then $[[A^{S,\leq}]] \cong eT^Se$ has stable range one. By Theorem 2.10, $A$ has stable range one. Analogously, $B$ has stable range one.

Conversely, we assume that $A$ and $B$ both have stable range one. Then by [6, Theorem 1], $T$ has stable range one. Thus we complete the proof by Theorem 2.2 and Lemma 2.10.

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