Journal of Mathematical Research & Exposition
Jan., 2009, Vol. 29, No. 1, pp. 137–145
DOI:10.3770/j.issn:1000-341X.2009.01.018
Http://jmre.dlut.edu.cn

The Hardy Spaces Estimates for the Commutator of Marcinkiewicz Integral

WANG Hong Bin, LIU Zong Guang
(Department of Mathematics, China University of Mining and Technology (Beijing), Beijing 100083, China)
(E-mail: hbwang_2006@163.com; liuzg@cumtb.edu.cn)

Abstract In this paper, we obtain the $(H^1, L^{n/(n-\beta)})$ and $(H\dot{K}_{q_1}^{n(1-1/q_1),p}, \dot{K}_{q_2}^{n(1-1/q_1),p})$ type estimates for the commutator of Marcinkiewicz integral with the kernel satisfying the logarithmic type Lipschitz conditions.

Keywords commutator; Marcinkiewicz integral; Lipschitz space; Hardy space; Herz type Hardy space.

Document code A
MR(2000) Subject Classification 42B20; 42B30
Chinese Library Classification O174.2

1. Introduction

Suppose that S^{n-1} is the unit sphere of R^n $(n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^1(\mathrm{R}^n)$ be homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein[9] introduced the Marcinkiewicz integral on R^n as follows

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t^3}\Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

And he proved that μ_Ω is of type (p, p) for $1 < p \leq 2$ and of weak type (1, 1) if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$. In [1], Benedek et al. proved that μ_Ω is of type (p, p) for $1 < p < \infty$ if $\Omega \in C^1(S^{n-1})$. Recently, Lee and Rim[6] established the (H^1, L^1), (L^∞, BMO) and (L^p, L^p) type boundedness of Marcinkiewicz integral μ_Ω when Ω satisfies a class of logarithmic type Lipschitz conditions. The main result in [6] is following theorem.

Theorem A[6] *Let $n\geq 2$ and $\Omega \in L^\infty(S^{n-1})$ be a homogeneous function of degree zero with*

Received date: 2006-12-18; **Accepted date**: 2007-09-04
Foundation item: the National Natural Science Foundation of China (No. 10371080); the Key Program Foundation of Ministry of Education of China (No. 108136).

cancellation property

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0.$$

Suppose, in addition, that there exist constants $C > 0$ *and* $\rho > 1$ *suth that*

$$|\Omega(y_1) - \Omega(y_2)| \le \frac{C}{\left(\log \frac{1}{|y_1 - y_2|}\right)^{\rho}}$$

hold uniformly in $y_1, y_2 \in S^{n-1}$. *Then the following inequalities hold:*

$$\|\mu_\Omega(f)\|_{L^1} \le C_1 \|f\|_{H^1}, \quad f \in H^1(\mathrm{R}^n),$$

$$\|\mu_\Omega(f)\|_{\mathrm{BMO}} \le C_\infty \|f\|_{L^\infty}, \quad f \in L^2 \cap L^\infty$$

and

$$\|\mu_\Omega(f)\|_{L^p} \le C_p \|f\|_{L^p}, \quad f \in L^p(\mathrm{R}^n),$$

where $1 < p < \infty$.

Remark 1 Since $0 \le |y_1 - y_2| \le 2$ for $y_1,\ y_2 \in \mathrm{S}^{n-1}$, it is reasonable that the above logarithmic type Lipschitz condition would be

$$|\Omega(y_1) - \Omega(y_2)| \le \frac{C}{\left(\log \frac{2}{|y_1 - y_2|}\right)^{\rho}}.$$

Let $b \in L^1_{\mathrm{loc}}(\mathrm{R}^n)$. The commutator generated by the Marcinkiewicz integral μ_Ω and b is defined by

$$\mu^b_\Omega(f)(x) = \left(\int_0^\infty \Big| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x) - b(y)]f(y)\mathrm{d}y\Big|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Let $0 < \beta \le 1$. The Lipschitz class $\mathrm{Lip}_\beta(\mathrm{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathrm{R}^n) = \left\{f : \|f\|_{\mathrm{Lip}_\beta} = \sup_{x,y\in\mathrm{R}^n; x\ne y} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty\right\}.$$

When $b \in \mathrm{Lip}_\beta(\mathrm{R}^n)$, Mo[8] proved the (L^p, L^r) type boundedness of high-order Marcinkiewicz integral commutator. The special case of the main result in [8] is the following theorem:

Theorem B[8] *Let* $b \in \mathrm{Lip}_\beta(\mathrm{R}^n)$, $0 < \beta < 1$. *If* $\Omega \in L^q(S^{n-1})$ *for some* $q \ge n/(n-\beta)$ *and* $\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0$, *then* μ^b_Ω *is bounded from* $L^p(\mathrm{R}^n)$ *to* $L^r(\mathrm{R}^n)$, *where* $1 < p < n/\beta$ *and* $1/r = 1/p - \beta/n$.

Inspired by [6] and [8], we establish the $(H^1, L^{n/(n-\beta)})$ and $(H\dot{K}^{n(1-1/q_1),p}_{q_1}, \dot{K}^{n(1-1/q_1),p}_{q_2})$ type boundedness of the Marcinkiewicz integral commutator μ^b_Ω, where Ω satisfies the logarithmic type Lipschitz conditions. To state our main result, we introduce the following definitions and auxiliary results.

Definition 1[10] *A function* $a(x)$ *on* R^n *is said to be an* H^1 *atom, if there exists a ball* B, *such that*

1) $\mathrm{supp}\, a \subset B$; *2)* $\|a\|_{L^\infty} \le |B|^{-1}$; *3)* $\int_{\mathrm{R}^n} a(x)\mathrm{d}x = 0$.

It is said that $f \in L^1(\mathrm{R}^n)$ belongs to Hardy space $H^1(\mathrm{R}^n)$, if $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ in the sense of distributions, where each a_j is an H^1 atom, $\lambda_j \in \mathbf{C}$ and $\sum_{j=-\infty}^{\infty} |\lambda_j| < \infty$. Furthermore, the $H^1(\mathrm{R}^n)$ seminorm is defined as

$$\|f\|_{H^1} = \inf \sum_{j=-\infty}^{\infty} |\lambda_j|,$$

where the infimum is taken over all above decompositions of f.

Remark 2 Let $a(x)$ be an H^1 atom. Then for any $p_0 \in [1, \infty]$, we have $\|a\|_{L^{p_0}} \le |B|^{-1+1/p_0}$.

Definition 2[7] *Let $B_k = \{x \in \mathrm{R}^n : |x| < 2^k\}$, $C_k = B_k \setminus B_{k-1}$ and χ_k be the characteristic function of the set C_k for $k \in \mathbf{Z}$. For $\alpha \in \mathbf{R}$, $0 < p < \infty$ and $0 < q < \infty$, the homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathrm{R}^n)$ is defined by*

$$\dot{K}_q^{\alpha,p}(\mathrm{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathrm{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \Big\}^{1/p}$$

with usual modification made with $p = \infty$.

Definition 3[7] *Let $\alpha \in \mathbf{R}$, $0 < p < \infty$ and $0 < q < \infty$. $G(f)$ denotes the Grand maximal function of $f \in \mathcal{S}'(\mathrm{R}^n)$. The Herz–type Hardy space $H\dot{K}_q^{\alpha,p}(\mathrm{R}^n)$ is defined by*

$$H\dot{K}_q^{\alpha,p}(\mathrm{R}^n) = \Big\{ f \in S'(\mathrm{R}^n) : G(f) \in \dot{K}_q^{\alpha,p}(\mathrm{R}^n) \Big\}$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \|G(f)\|_{\dot{K}_q^{\alpha,p}}.$$

Definition 4[7] *Let $1 < q < \infty$, $n(1 - 1/q) \le \alpha < \infty$, $s \in \mathbf{N}$ and $s \ge [\alpha + n(1/q - 1)]$. A function $a(x)$ on R^n is said to be a central $(\alpha, q)_s$ atom, if*

1) $\operatorname{supp} a \subset B(0, r) = \{x \in \mathrm{R}^n : |x| < r\}$ *for some $r > 0$,*
2) $\|a\|_{L^q} \le |B(0, r)|^{-\alpha/n}$,
3) $\int_{\mathrm{R}^n} a(x) x^\gamma \mathrm{d}x = 0$ *for any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathrm{N}^n$ with $0 \le |\gamma| = \sum_{i=1}^n \gamma_i \le s$.*

Theorem C[7] *Let $0 < p$, $q < \infty$, $n(1 - 1/q) \le \alpha < \infty$, $s \ge [\alpha + n(1/q - 1)]$. Then $f \in H\dot{K}_q^{\alpha,p}(\mathrm{R}^n)$ if and only if $f(x) = \sum_{i=-\infty}^{\infty} \lambda_i a_i(x)$, in the $\mathcal{S}'$ sense, where each a_i is a central $(\alpha, q)_s$ atom, supp $a_i \subset B_i$, $\lambda_i \in \mathbf{C}$ and $\sum_{i=-\infty}^{\infty} |\lambda_i|^p < \infty$. Furthermore,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \inf \Big(\sum_{i=-\infty}^{\infty} |\lambda_i|^p \Big)^{1/p},$$

where the infimum is taken over all above decompositions of f.

2. The main results and their proofs

Theorem 1 *Let $b \in \mathrm{Lip}_\beta(\mathrm{R}^n)$, $0 < \beta < 1$. If Ω is a homogeneous function of degree zero and satisfies the following conditions:*

(i) $\int_{S^{n-1}} \Omega(x') \mathrm{d}\sigma(x') = 0$ *and $\Omega \in L^q(S^{n-1})$ for some $q \ge n/(n - \beta)$;*

(ii) there exist constants $C>0$ *and* $\rho>1$*, such that*

$$|\Omega(y_1)-\Omega(y_2)|\le \frac{C}{(\log\frac{2}{|y_1-y_2|})^{\rho}}$$

for any $y_1, y_2 \in S^{n-1}$*. Then* μ_Ω^b *is bounded from* $H^1(\mathrm{R}^n)$ *to* $L^{n/(n-\beta)}(\mathrm{R}^n)$*.*

Theorem 2 *Let* $b\in \mathrm{Lip}_\beta(\mathrm{R}^n)$*,* $0<\beta<1$*,* $0<p<\infty$*,* $1<q_1$*,* $q_2<\infty$ *and* $1/q_2=1/q_1-\beta/n$*. If* Ω *is a homogeneous function of degree zero and satisfies the following conditions:*

(i) $\int_{S^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0$ *and* $\Omega\in L^q(S^{n-1})$ *for some* $q\ge\max\{q_2, n/(n-\beta)\}$*;*

(ii) there exist constants $C>0$ *and* $\rho>\max(1, 1/p)$ *such that*

$$|\Omega(y_1)-\Omega(y_2)|\le \frac{C}{(\log\frac{2}{|y_1-y_2|})^{\rho}}$$

for any $y_1, y_2 \in S^{n-1}$*. Then* μ_Ω^b *is bounded from* $H\dot{K}_{q_1}^{n(1-1/q_1),p}(\mathrm{R}^n)$ *to* $\dot{K}_{q_2}^{n(1-1/q_1),p}(\mathrm{R}^n)$*.*

Proof of Theorem 1 We need to prove that $\|\mu_\Omega^b(a)\|_{L^r}\le C$ for any H^1 atom a, where $r=n/(n-\beta)$ and supp $a\subset B(x_0,l)$.

$$\begin{aligned}\|\mu_\Omega^b(a)\|_{L^r}\le&\Big[\int_{2B}|\mu_\Omega^b(a)(x)|^r\mathrm{d}x\Big]^{1/r}+\Big[\int_{(2B)^c}|\mu_\Omega^b(a)(x)|^r\mathrm{d}x\Big]^{1/r}\\ \le&\Big[\int_{2B}|\mu_\Omega^b(a)(x)|^r\mathrm{d}x\Big]^{1/r}+\\ &\Big\{\int_{(2B)^c}\Big[\int_0^{|x-x_0|+2l}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{r/2}\mathrm{d}x\Big\}^{1/r}+\\ &\Big\{\int_{(2B)^c}\Big[\int_{|x-x_0|+2l}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{r/2}\mathrm{d}x\Big\}^{1/r}\\ =:&I_1+I_2+I_3.\end{aligned}$$

Choose p_1 and s_1 such that $1<p_1<n/\beta$ and $1/s_1=1/p_1-\beta/n$. It is easy to see that $r<s_1$, by Hölder's inquality and Theorem B, we get

$$\begin{aligned}I_1&\le C\|\mu_\Omega^b(a)\|_{L^{s_1}}|2B|^{1/r-1/s_1}\le C\|a\|_{L^{p_1}}|2B|^{1/r-1/s_1}\\ &\le C|B|^{-1+1/p_1}|B|^{1/r-1/s_1}\le C.\end{aligned}$$

Using the fact that $|x-y|\sim|x-x_0|\sim|x-x_0|+2l$ for any $y\in B$ and $x\in(2B)^c$, Minkowski's inequality and Hölder's inquality, we have

$$\begin{aligned}I_2&\le C\Big\{\int_{(2B)^c}\Big[\int_{R^n}\Big(\int_{|x-y|}^{|x-x_0|+2l}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\frac{|\Omega(x-y)||a(y)|}{|x-y|^{n-1}}|b(x)-b(y)|\mathrm{d}y\Big]^r\mathrm{d}x\Big\}^{1/r}\\ &\le C\int_B\Big\{\int_{(2B)^c}\Big[\frac{l^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}|b(x)-b(y)|\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\ &\le C\int_B\sum_{k=1}^{\infty}\Big\{\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big[\frac{l^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}|b(x)-b(y)|\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\ &\le C\int_B\sum_{k=1}^{\infty}2^{-k/2}(2^kl)^{-n}(2^{k+1}l)^{\beta}\|b\|_{\mathrm{Lip}_\beta}\Big[\int_{|x-x_0|<2^{k+1}l}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}|a(y)|\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\le C\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}\int_B\sum_{k=1}^{\infty}2^{-k/2}(2^kl)^{-n}(2^{k+1}l)^{\beta}(2^{k+1}l)^{n/r}|a(y)|\mathrm{d}y\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}\int_B\sum_{k=1}^{\infty}2^{-k[(1/2-\beta)+n(1-1/r)]}|a(y)|\mathrm{d}y\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}\|a\|_{L^1}\le C\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}.
\end{aligned}
$$

In the above last inequality, we applied the fact that $-n+\beta+n/r=0$ and the series is convergent.

Let us turn to estimate I_3 now. Note that $t\ge|x-x_0|+2l\ge|x-x_0|+|y-x_0|\ge|x-y|$ for any $y\in B$ and the cancellation property of a,

$$
\begin{aligned}
&\Big[\int_{|x-x_0|+2l}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\\
&=\Big[\int_{|x-x_0|+2l}^{\infty}\Big|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\\
&=\Big|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|\Big(\int_{|x-x_0|+2l}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\
&=C\Big|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a(y)\mathrm{d}y\Big|\frac{1}{|x-x_0|+2l}\\
&\le C\int_B|b(x)-b(x_0)|\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|a(y)|}{|x-x_0|+2l}\mathrm{d}y+\\
&\quad C\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(y)-b(x_0)|\,|a(y)|}{|x-x_0|+2l}\mathrm{d}y.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
I_3\le&C\Big\{\Big[\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}+\\
&\Big[\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(y)-b(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}\Big\}\\
=:&C(I_{31}+I_{32}).
\end{aligned}
$$

Applying the estimate

$$
\begin{aligned}
\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|&\le\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-y|^{n-1}}\Big|+\Big|\frac{\Omega(x-x_0)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\\
&\le\frac{C(1+|\Omega(x-x_0)|)}{|x-x_0|^{n-1}(\log\frac{|x-x_0|}{l})^{\rho}},
\end{aligned}
$$

Minkowski's inequality and Hölder's inequality, we get

$$
\begin{aligned}
I_{31}&\le\int_B\Big\{\int_{(2B)^c}\Big[\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le\int_B\Big\{\sum_{k=1}^{\infty}\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big[\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{|b(x)-b(x_0)|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^{\infty}(2^kl)^{\beta-1}(2^{k+1}l)^{n(1/r-1/q)}\|b\|_{\mathrm{Lip}_\beta}\times
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\int_{2^k l\le|x-x_0|<2^{k+1}l}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^q\mathrm{d}x\Big]^{1/q}|a(y)|\mathrm{d}y\\
\le& C\int_B\sum_{k=1}^{\infty}(2^kl)^{\beta-1}(2^{k+1}l)^{n(1/r-1/q)}\|b\|_{\mathrm{Lip}_\beta}(2^kl)^{-(n-1)}(\log 2^k)^{-\rho}(2^kl)^{n/q}(\|\Omega\|_{L^q(S^{n-1})}+1)|a(y)|\mathrm{d}y\\
\le& C\sum_{k=1}^{\infty}(2^kl)^{\beta-1}(2^{k+1}l)^{n(1/r-1/q)}\|b\|_{\mathrm{Lip}_\beta}(2^kl)^{-(n-1)}(\log 2^k)^{-\rho}(2^kl)^{n/q}(\|\Omega\|_{L^q(S^{n-1})}+1)\\
\le& C\|b\|_{\mathrm{Lip}_\beta}(\|\Omega\|_{L^q(S^{n-1})}+1)\sum_{k=1}^{\infty}k^{-\rho}\le C\|b\|_{\mathrm{Lip}_\beta}(\|\Omega\|_{L^q(S^{n-1})}+1).
\end{aligned}
$$

It is easy to see that $|x-y|\sim|x-x_0|+2l$ for any $y\in B$ and $x\in(2B)^c$. By Minkowski's inequality, we have

$$
\begin{aligned}
I_{32}&\le C\int_B\Big\{\int_{(2B)^c}\Big[\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{|b(y)-b(x_0)|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^{\infty}l^{\beta}(2^kl)^{-n}\|b\|_{\mathrm{Lip}_\beta}\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^{\infty}2^{-kn(1-1/r)}|a(y)|\mathrm{d}y\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}\\
&\le C\|a\|_{L^1}\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}\le C\|b\|_{\mathrm{Lip}_\beta}\|\Omega\|_{L^q(S^{n-1})}.
\end{aligned}
$$

Thus, we can see that

$$I_3\le C(I_{31}+I_{32})\le C\|b\|_{\mathrm{Lip}_\beta}(\|\Omega\|_{L^q(S^{n-1})}+1).$$

Combining the estimates for I_1, I_2 and I_3, we obtain that

$$\|\mu_\Omega^b(a)\|_{L^{n/(n-\beta)}}\le C\|b\|_{\mathrm{Lip}_\beta}(\|\Omega\|_{L^q(S^{n-1})}+1).$$

Proof of Theorem 2 Let $f\in H\dot{K}_{q_1}^{\alpha,p}(\mathrm{R}^n)$, where $\alpha=n(1-1/q_1)$. We have that $f=\sum_{i=-\infty}^{\infty}\lambda_ia_i$, where each a_i is a central $(\alpha,q_1)_s$ atom, supp $a_i\subset B_i$ and $\sum_{i=-\infty}^{\infty}|\lambda_i|^p<\infty$. Writting

$$D_i^{(1)}(x)=\Big[\int_0^{|x|+2^{i+1}}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_i(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2},$$

$$D_i^{(2)}(x)=\Big[\int_{|x|+2^{i+1}}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}(b(x)-b(y))a_i(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2},$$

we have

$$
\begin{aligned}
\|\mu_\Omega^b(f)\|_{\dot{K}_{q_2}^{\alpha,p}}^p\le& C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=k-1}^{\infty}|\lambda_i|\|\mu_\Omega^b(a_i)\chi_k\|_{L^{q_2}}\Big)^p+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=-\infty}^{k-2}|\lambda_i|\|\mu_\Omega^b(a_i)\chi_k\|_{L^{q_2}}\Big)^p\\
\le& C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=k-1}^{\infty}|\lambda_i|\|\mu_\Omega^b(a_i)\chi_k\|_{L^{q_2}}\Big)^p+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=-\infty}^{k-2}|\lambda_i|\|D_i^{(1)}\chi_k\|_{L^{q_2}}\Big)^p+\\
&C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=-\infty}^{k-2}|\lambda_i|\|D_i^{(2)}\chi_k\|_{L^{q_2}}\Big)^p
\end{aligned}
$$

$$=:C(J_1+J_2+J_3).$$

By Theorem B and the size condition of a_i, we get

$$\begin{aligned}
J_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum_{i=k-1}^{\infty}|\lambda_i|\|a_i\|_{L^{q_1}}\Big)^p\\
&\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{i=k-1}^{\infty}|\lambda_i|2^{(k-i)\alpha}\Big)^p\\
&\le C\begin{cases}\displaystyle\sum_{k=-\infty}^{\infty}\sum_{i=k-1}^{\infty}|\lambda_i|^p2^{(k-i)\alpha p}, & 0<p\le 1\\ \displaystyle\sum_{k=-\infty}^{\infty}(\sum_{i=k-1}^{\infty}|\lambda_i|^p2^{(k-i)\alpha p})(\sum_{i=k-1}^{\infty}2^{(k-i)\alpha p/2})^{p/p'}, & 1<p<\infty\end{cases} \qquad (1)\\
&\le C\begin{cases}\displaystyle\sum_{i=-\infty}^{\infty}|\lambda_i|^p\sum_{k=-\infty}^{i+1}2^{(k-i)\alpha p}, & 0<p\le 1\\ \displaystyle\sum_{i=-\infty}^{\infty}|\lambda_i|^p\sum_{k=-\infty}^{i+1}2^{(k-i)\alpha p/2}, & 1<p<\infty\end{cases}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p.
\end{aligned}$$

Applying the fact that $|x-y|\sim|x|\sim|x|+2^{i+1}$ for any $x\in C_k$ and $y\in B_i$ with $k\ge i+2$ and Minkowski's inequality, we have

$$\begin{aligned}
\|D_i^{(1)}\chi_k\|_{L^{q_2}} &\le\Big\{\int_{C_k}\Big[\int_{R^n}\Big(\int_{|x-y|}^{|x|+2^{i+1}}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\frac{|\Omega(x-y)||a_i(y)|}{|x-y|^{n-1}}|b(x)-b(y)|\mathrm{d}y\Big]^{q_2}\mathrm{d}x\Big\}^{1/q_2}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\Big\{\int_{C_k}\Big[\int_{B_i}\frac{2^{i/2}|\Omega(x-y)||a_i(y)|}{|x-y|^{n+1/2-\beta}}\mathrm{d}y\Big]^{q_2}\mathrm{d}x\Big\}^{1/q_2}\\
&\le C2^{i/2}2^{-k(n+1/2-\beta)}\|b\|_{\mathrm{Lip}_\beta}\int_{B_i}\Big[\int_{C_k}|\Omega(x-y)|^{q_2}\mathrm{d}x\Big]^{1/q_2}|a_i(y)|\mathrm{d}y.
\end{aligned}$$

When $i\le k-2$, we note that $|x-y|\le 2^{k+1}$ for any $x\in C_k$ and $y\in B_i$. By Hölder's inequality, we have

$$\Big[\int_{C_k}|\Omega(x-y)|^{q_2}\mathrm{d}x\Big]^{1/q_2}\le C2^{kn/q_2}\|\Omega\|_{L^q(S^{n-1})}, \qquad (2)$$

and

$$\|a_i\|_{L^1}\le\|a_i\|_{L^{q_1}}|B_i|^{1-1/q_1}\le C. \qquad (3)$$

These estimates follow that

$$\begin{aligned}
\|D_i^{(1)}\chi_k\|_{L^{q_2}} &\le C2^{i/2}2^{-k(n+1/2-\beta)}2^{kn/q_2}\|\Omega\|_{L^q(S^{n-1})}\|a_i\|_{L^1}\|b\|_{\mathrm{Lip}_\beta}\\
&\le C2^{(i-k)[n(1-1/q_1)+1/2]}2^{-i\alpha}\|\Omega\|_{L^q(S^{n-1})}\|b\|_{\mathrm{Lip}_\beta}.
\end{aligned}$$

It follows that

$$\begin{aligned}
J_2 &\le C\sum_{k=-\infty}^{\infty}\Big(\sum_{i=-\infty}^{k-2}|\lambda_i|2^{(i-k)[n(1-1/q_1)+1/2-\alpha]}\Big)^p\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta}\\
&\le C\sum_{i=-\infty}^{\infty}|\lambda_i|^p\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta}.
\end{aligned}$$

Let us turn to estimate J_3 now. By the fact that $t \geq |x| + 2^{i+1} \geq |x| + |y| \geq |x-y|$ for any $y \in B_i$ with $i \leq k-2$ and the cancelation property of a_i, we get

$$\begin{aligned} D_i^{(2)}(x) =& \Big[\int_{|x|+2^{i+1}}^{\infty} \Big| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) a_i(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big]^{1/2} \\ =& \Big[\int_{|x|+2^{i+1}}^{\infty} \Big| \int_{B_i} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) a_i(y) \mathrm{d}y \Big|^2 \frac{\mathrm{d}t}{t^3} \Big]^{1/2} \\ =& \Big| \int_{B_i} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) a_i(y) \mathrm{d}y \Big| \Big(\int_{|x|+2^{i+1}}^{\infty} \frac{\mathrm{d}t}{t^3} \Big)^{1/2} \\ \leq& \Big| \int_{B_i} (b(x)-b(0)) \Big[\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big] \frac{a_i(y)}{|x|+2^{i+1}} \mathrm{d}y \Big| + \\ & \Big| \int_{B_i} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{(b(y)-b(0)) a_i(y)}{|x|+2^{i+1}} \mathrm{d}y \Big|. \end{aligned}$$

Thus

$$\begin{aligned} \|D_i^{(2)} \chi_k\|_{L^{q_2}} \leq& C \Big\{ \Big[\int_{C_k} \Big(\int_{B_i} \Big| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big| \frac{|b(x)-b(0)||a_i(y)|}{|x|+2^{i+1}} \mathrm{d}y \Big)^{q_2} \mathrm{d}x \Big]^{1/q_2} + \\ & \Big[\int_{C_k} \Big(\int_{B_i} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} \frac{|b(y)-b(0)||a_i(y)|}{|x|+2^{i+1}} \mathrm{d}y \Big)^{q_2} \mathrm{d}x \Big]^{1/q_2} \Big\} \\ =:& C(E_1+E_2), \end{aligned}$$

$$\begin{aligned} E_1 =& \Big[\int_{C_k} \Big(\int_{B_i} \Big| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big| \frac{|b(x)-b(0)||a_i(y)|}{|x|+2^{i+1}} \mathrm{d}y \Big)^{q_2} \mathrm{d}x \Big]^{1/q_2} \\ \leq& \int_{B_i} \Big\{ \int_{C_k} \Big[\Big| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big| \frac{|b(x)-b(0)|}{|x|+2^{i+1}} \mathrm{d}y \Big]^{q_2} \mathrm{d}x \Big\}^{1/q_2} |a_i(y)| \mathrm{d}y \\ \leq& C\|b\|_{\mathrm{Lip}_\beta} 2^{-k(1-\beta)} \int_{B_i} \Big[\int_{C_k} \Big| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big|^{q_2} \mathrm{d}x \Big]^{1/q_2} |a_i(y)| \mathrm{d}y \\ \leq& C\|b\|_{\mathrm{Lip}_\beta} 2^{-k(1-\beta)} 2^{kn(1/q_2-1/q_1)} \int_{B_i} \Big[\int_{C_k} \Big| \frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x)}{|x|^{n-1}} \Big|^{q_1} \mathrm{d}x \Big]^{1/q_1} |a_i(y)| \mathrm{d}y \\ \leq& C\|b\|_{\mathrm{Lip}_\beta} 2^{-k(1-\beta)} 2^{kn(1/q_2-1/q_1)} (2^{k-1})^{-(n-1)} (\log 2^{k-i})^{-\rho} 2^{kn/q_1} \|\Omega\|_{L^q(S^{n-1})} \int_{B_i} |a_i(y)| \mathrm{d}y \\ \leq& C\|b\|_{\mathrm{Lip}_\beta} 2^{kn(1/q_1-1)} (k-i)^{-\rho} \|\Omega\|_{L^q(S^{n-1})} \|a_i\|_{L^1} \\ \leq& C 2^{-kn(1-1/q_1)} (k-i)^{-\rho} \|\Omega\|_{L^q(S^{n-1})} \|b\|_{\mathrm{Lip}_\beta}. \end{aligned}$$

Using the fact that $|x-y| \sim |x|$ for any $x \in C_k$ and $y \in B_i$ with $i \leq k-2$, Minkowski's inequality, (2) and (3), we have

$$\begin{aligned} E_2 \leq& C \int_{B_i} \Big\{ \int_{C_k} \Big[\frac{|\Omega(x-y)|}{|x-y|^{n-1}} \frac{|b(y)-b(0)|}{|x|+2^{i+1}} \Big]^{q_2} \mathrm{d}x \Big\}^{1/q_2} |a_i(y)| \mathrm{d}y \\ \leq& C 2^{i\beta} 2^{-kn} \|b\|_{\mathrm{Lip}_\beta} \int_{B_i} \Big[\int_{C_k} |\Omega(x-y)|^{q_2} \mathrm{d}x \Big]^{1/q_2} |a_i(y)| \mathrm{d}y \\ \leq& C\|b\|_{\mathrm{Lip}_\beta} 2^{i\beta} 2^{-kn} 2^{kn/q_2} \|\Omega\|_{L^q(S^{n-1})} \\ \leq& C 2^{i\beta - kn + k(n/q_1 - \beta)} \|\Omega\|_{L^q(S^{n-1})} \|b\|_{\mathrm{Lip}_\beta}. \end{aligned}$$

Combining the estimates for E_1 and E_2, we get

$$\|D_i^{(2)}\chi_k\|_{L^{q_2}} \leq C2^{-kn(1-1/q_1)}(k-i)^{-\rho}\|\Omega\|_{L^q(S^{n-1})}\|b\|_{\mathrm{Lip}_\beta}.$$

When $0 < p \leq 1$, noticing $\alpha = n(1-1/q_1)$ and $\rho p > 1$, it follows that

$$\begin{aligned} J_3 &\leq C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|2^{-kn(1-1/q_1)}(k-i)^{-\rho}\Big\}^p\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta} \\ &\leq C\sum_{k=-\infty}^{\infty}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|^p(k-i)^{-\rho p}\Big\}\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta} \\ &\leq C\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta}\sum_{i=-\infty}^{\infty}|\lambda_i|^p. \end{aligned}$$

When $p > 1$, by Hölder's inequality and the fact that $\rho > 1$, we obtain that

$$\begin{aligned} J_3 &\leq C\sum_{k=-\infty}^{\infty}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|(k-i)^{-\rho(1/p+1/p')}\Big\}^p\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta} \\ &\leq C\sum_{k=-\infty}^{\infty}\Big\{\sum_{i=-\infty}^{k-2}|\lambda_i|^p(k-i)^{-\rho}\Big\}\Big\{\sum_{i=-\infty}^{k-2}(k-i)^{-\rho}\Big\}^{p/p'}\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta} \\ &\leq C\|\Omega\|^p_{L^q(S^{n-1})}\|b\|^p_{\mathrm{Lip}_\beta}\sum_{i=-\infty}^{\infty}|\lambda_i|^p. \end{aligned}$$

Combining the estimates for J_1, J_2 and J_3, we get

$$\|\mu^b_\Omega(f)\|_{\dot K^{\alpha,p}_{q_2}} \leq C\|\Omega\|_{L^q(S^{n-1})}\|b\|_{\mathrm{Lip}_\beta}\|f\|_{H\dot K^{\alpha,p}_{q_1}}.$$

References

[1] BENEDEK A, CALDERÓN A P, PANZONE R. *Convolution operators on Banach space valued functions* [J]. Proc. Nat. Acad. Sci. U.S.A., 1962, **48**: 356–365.

[2] CHEN Jiecheng, FAN Dashan, PAN Yibiao. *A note on a Marcinkiewicz integral operator* [J]. Math. Nachr., 2001, **227**: 33–42.

[3] CHEN Dongxiang, ZHANG Pu, CHEN Jiecheng. *Boundedness of higher-order commutators for Marcinkiewicz integrals with homogeneous kernel on Herz-Hardy spaces* [J]. Appl. Math. J. Chinese Univ. Ser. A, 2004, **19**(1): 109–117. (in Chinese)

[4] DING Yong, LU Shanzhen, XUE Qingying. *Marcinkiewicz integral on Hardy spaces* [J]. Integral Equations Operator Theory, 2002, **42**(2): 174–182.

[5] DING Yong, LU Shanzhen, ZHANG Pu. *Weighted weak type estimates for commutators of the Marcinkiewicz integrals* [J]. Sci. China Ser. A, 2004, **47**(1): 83–95.

[6] LEE J, RIM K S. *Estimates of Marcinkiewicz integrals with bounded homogeneous kernels of degree zero* [J]. Integral Equations Operator Theory, 2004, **48**(2): 213–223.

[7] LU Shanzhen, YANG Dachun. *The weighted Herz-type Hardy space and its applications* [J]. Sci. China Ser. A, 1995, **38**(6): 662–673.

[8] MO Huixia. *Boundedness of Some Operators and Their Commutators* [D]. Doctor Degree Dissertation, Beijing Normal University, 2006.

[9] STEIN E M. *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz* [J]. Trans. Amer. Math. Soc., 1958, **88**: 430–466.

[10] STEIN E M. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* [M]. Princeton University Press, Princeton, NJ, 1993.

[11] TORCHINSKY A, WANG Shilin. *A note on the Marcinkiewicz integral* [J]. Colloq. Math., 1990, **60/61**(1): 235–243.