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The Existence and Non-Existence of Global Solutions for a Nonlinear Wave Equation

WANG Yan Ping
(Department of Mathematics and Physics, Zhengzhou Institute of Aeronautical Industry Management, Henan 450015, China)
(E-mail: wangyping1968@163.com)

Abstract This paper deals with the existence and uniqueness of the global solution of the initial boundary value problem of a class of wave equation. In the meantime, it gives the sufficient conditions of blow-up of the solution for the problem in finite time.

Keywords nonlinear wave equation; initial boundary value problem; global solution; blow-up of solution.

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1. Introduction

In this paper, we will study the existence and the uniqueness of the global generalized solution and the global classical solution and the blow-up of the solution to the following initial boundary value problem for the nonlinear wave equation

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = g(u_x)_x,\ x \in (0,l),\ t \in (0,T), \tag{1.1}$$

$$u_x(0,t) = u_x(l,t) = 0,\ u_{xxx}(0,t) = u_{xxx}(l,t) = 0,\ t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in [0,l], \tag{1.3}$$

where $u(x,t)$ denotes the unknown function, $a > 0,\ b > 0$ are two constants, $g(s)$ is the given nonlinear function, $u_0(x)$ and $u_1(x)$ are given initial value functions and satisfy the boundary condition (1.2). The subscripts t and x indicate the partial derivative with respect to t and x.

There are several examples of physical problems, which can be formulated as equation (1.1).

In the study of a weakly nonlinear analysis of elasto-plastic-microstructure models for a longitudinal motion of an elasto-plastic bar, the following nonlinear partial differential equation

$$u_{tt} + u_{xxxx} = a(u_x^2)_x \tag{1.4}$$

is given[1], where $a \neq 0$ is constant.

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In [2, 3], for the study of strain solitary waves in nonlinear rods, a longitudinal wave equation reads

$$u_{tt} - [a_0 + na_1(u_x)^{n-1}]u_{xx} - a_2 u_{xxtt} = 0, \tag{1.5}$$

here $a_0,\ a_2$ are constants, a_1 is an arbitrary real number, n is a natural number.

The papers [4, 5] studied the dynamics of dense lattices, and gave the equation

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = (u_x^2)_x. \tag{1.6}$$

Obviously, equations (1.4)–(1.6) are the special cases of equation (1.1).

For the equation (1.4), the author in [6] studied the existence and non-existence of global solutions for the initial boundary value problem.

In [7], the authors proved the existence and uniqueness of classical global solution and blow-up of non-global solution to the initial boundary value problem for the equation (1.5).

However, we have not seen any discussion on the initial boundary value problem for equation (1.6).

As for equation (1.1), as the generalized case of equation (1.6), there have not been any results. The aim of the present paper is to prove that under certain conditions, the problem (1.1)–(1.3) has a unique global generalized and classical solutions, and to give sufficient conditions of the nonexistence of global solutions to the problem (1.1)–(1.3). Moreover, as application of our abstract theorem, we shall prove that the problem (1.2), (1.3) and (1.6) do not possess global generalized and classical solutions under certain conditions.

The general method is to establish a differential inequality of energy of solution in order to get the blowup result of a solution of a nonlinear evolution equation[8−11]. To prove the blow-up of solution by the "concavity method", we will construct a differential inequality (3.3) and by the aid of the inequality we shall complete the related proof.

This paper is organized as follows: In Section 2, we prove the existence and uniqueness of global generalized and classical solutions of the problem (1.1)–(1.3). The nonexistence of global solutions to the problem (1.1)–(1.3) is discussed in Section 3. In Section 4, we prove the nonexistence of global solutions to the problem (1.2), (1.3) and (1.6).

2. Global solution of the problem (1.1)–(1.3)

For the problem (1.1)–(1.3), we have the following Theorems 2.1 and 2.2.

Theorem 2.1 *Suppose that $g \in C^2(R)$ and there is a constant γ such that $g'(s) \geq \gamma$ for any $s \in R$, $u_0(x) \in H^4[0,1],\ u_1(x) \in H^3[0,1]$ and $u_0(x),\ u_1(x)$ satisfy the boundary conditions (1.2). Then the problem (1.1)–(1.3) has a unique generalized global solution*

$$u(x,t) \in C([0,T];\ H^4(0,l)) \cap C^1([0,T];\ H^3(0,l)) \cap C^2([0,T];\ H^2(0,l)).$$

Theorem 2.2 *Suppose that the conditions of Theorem 2.1 hold. If $g \in C^3(R)$ and $g''(0) = 0$, $u_0(x) \in H^6[0,1],\ u_1(x) \in H^5[0,1]$. Then the problem (1.1)–(1.3) has a unique global classical solution*

$$u(x,t) \in C([0,T];\ C^4[0,l]) \cap C^1([0,T];\ C^3[0,l]) \cap C^2([0,T];\ C^2[0,l]).$$

The above two Theorems can be proved in the same method as used in [12].

3. Nonexistence of global solutions of the problem (1.1)–(1.3)

To discuss the blow-up of the solution, we need the following lemma:

Lemma 3.1 (Jensen inequality) *Let $g(x)$ be defined on (a,b), $g(x)\in[a_1,\ b_1]$, where $a,\ b,\ a_1,\ b_1\le\infty$, $f(s)$ is a continuous and convex function on (a_1,b_1), $q(x)\in L^1[a,b]$, and $q(x)\ge 0$. Then it follows that*

$$f\Big(\frac{\int_a^b g(x)q(x)\mathrm{d}x}{\int_a^b q(x)\mathrm{d}x}\Big)\le\frac{\int_a^b f(g(x))q(x)\mathrm{d}x}{\int_a^b q(x)\mathrm{d}x}$$

when the right side is finite.

Theorem 3.2 *Suppose that the following conditions hold.*

(1) $g(s)$ is a convex function, $g(0)=0$, $g(s)\ge\delta s^q$, where $\delta>0$ is a real number and $q>1$ is an even number.

(2) $\frac{\pi}{2l}\int_0^l u_0(x)\cos\frac{\pi x}{l}\mathrm{d}x=\alpha\ge\{\frac{\pi^2l^2+a\pi^4}{l^4}(\frac{l}{\pi})^{q+1}\delta^{-1}\}^{\frac{1}{q-1}},\ \frac{\pi}{2l}\int_0^l u_1(x)\cos\frac{\pi x}{l}\mathrm{d}x=\beta>0$. Then the solution $u(x,t)$ of the problem (1.1)–(1.3) blows up in finite time T_0, i.e.,

$$\lim_{t\to T_0^-}\sup_{x\in(0,l)}|u(\cdot,t)|\to+\infty.$$

Proof Let

$$\varphi(t)=\frac{\pi}{2l}\int_0^l u(x,t)\cos\frac{\pi x}{l}\mathrm{d}x.$$

Multiplying both sides of equation (1.1) by $\frac{\pi}{2l}\cos\frac{\pi x}{l}$ and integrating by parts, we obtain

$$(1+\frac{b\pi^2}{l^2})\ddot\varphi+(\frac{\pi^2}{l^2}+\frac{a\pi^4}{l^4})\varphi=\frac{\pi}{2l}\int_0^l g(u_x)_x\cos\frac{\pi x}{l}\mathrm{d}x,\tag{3.1}$$

where and in the sequel “ $\cdot$ ” denotes the derivative with respect to t.

Since $f(s)$ is even and convex, we have by using integration by parts and the Jensen inequality that

$$\begin{aligned}\frac{\pi}{2l}\int_0^l g(u_x)_x\cos\frac{\pi x}{l}\mathrm{d}x&=\frac{\pi^2}{2l^2}\int_0^l g(u_x)\sin\frac{\pi x}{l}\mathrm{d}x\\&\ge\frac{\pi}{l}g(-\frac{\pi^2}{2l^2}\int_0^l u(x,t)\cos\frac{\pi x}{l}\mathrm{d}x)\\&\ge\delta(\frac{\pi}{l})^{q+1}\varphi(t)^q,\ t>0.\end{aligned}\tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$(1+\frac{b\pi^2}{l^2})\ddot\varphi(t)+(\frac{\pi^2}{l^2}+\frac{a\pi^4}{l^4})\varphi(t)\ge\delta(\frac{\pi}{l})^{q+1}\varphi(t)^q,\ t>0\tag{3.3}$$

with $\varphi(0)=\alpha>0$ and $\dot\varphi(0)=\beta>0$.

Since $\varphi(0)=\alpha>0,\ \dot\varphi(0)=\beta>0$, from the continuity of $\varphi(t)$ it follows that there is a right neighborhood $(0,\ \rho)$ of the point t, in which $\dot\varphi(t)>0$, hence $\varphi(t)>\varphi(0)>0$.

Now, we prove $\dot\varphi(t)>0$ for any $t>0$. Suppose that this result is false. Then there is $t_0>0$, such that when $0<t<t_0$, $\dot\varphi(t)>0$, but $\dot\varphi(t_0)=0$, then $\varphi(t)$ is monotonically increasing on

$[0,\ t_0)$, i.e. $\varphi(t) \geq \alpha$, $t \in [0, t_0]$. It follows from (3.3) that on $(0,\ t_0]$

$$\begin{aligned}\ddot{\varphi}(t) &\geq \frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}\varphi(t)\Big[\frac{l^4}{l^2\pi^2 + a\pi^4}\delta(\frac{\pi}{l})^{q+1}(\varphi(t))^{q-1} - 1\Big] \\ &\geq \frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}\alpha\Big[\frac{l^4}{l^2\pi^2 + a\pi^4}\delta(\frac{\pi}{l})^{q+1}\alpha^{q-1} - 1\Big] \geq 0.\end{aligned}$$

Therefore, $\dot{\varphi}(t)$ is monotonically increasing on $[0, t_0]$. This is a contradiction with $\dot{\varphi}(t_0) = 0$. This shows that $\dot{\varphi}(t) > 0$ and $\varphi(t) > \varphi(0)$ for any $t > 0$. Multiplying both sides of (3.3) by $2\dot{\varphi}(t)$ and integrating the product on $[0,\ t]$, we see

$$\begin{aligned}(\dot{\varphi}(t))^2 &\geq \beta^2 - \frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}\left(\varphi(t)^2 - \alpha^2\right) + \frac{l^2}{l^2 + b\pi^2}\frac{2\delta}{q+1}(\frac{\pi}{l})^{q+1}\left(\varphi(t)^{q+1} - \alpha^{q+1}\right) \\ &= J(\varphi(t)).\end{aligned} \tag{3.4}$$

Obviously $J(\alpha) = \beta^2 > 0$, and

$$\begin{aligned}J'(\varphi(t)) &= 2\delta\frac{l^2}{l^2 + b\pi^2}(\frac{\pi}{l})^{q+1}\varphi(t)^q - 2\frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}\varphi(t) \\ &> 2\frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}\alpha\Big(\frac{\delta l^4}{l^2\pi^2 + a\pi^4}(\frac{\pi}{l})^{q+1}\alpha^{q-1} - 1\Big) \geq 0.\end{aligned}$$

It is easy to know that $J(\varphi(t)) > J(\varphi(0)) = J(\alpha) > 0,\ t > 0$.

Extracting the square root of both sides of (3.4) we obtain

$$\dot{\varphi}(t) \geq \Big[\frac{l^2}{l^2 + b\pi^2}\frac{2\delta}{q+1}(\frac{\pi}{l})^{q+1}(\varphi(t)^{q+1} - \alpha^{q+1}) - \frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}(\varphi(t)^2 - \alpha^2) + \beta^2\Big]^{\frac{1}{2}},\ t > 0$$

which implies that the interval $[0,\ T_1)$ of the existence of $\varphi(t)$ is finite, i.e.,

$$T_1 \leq \int_\alpha^{+\infty}\Big[\frac{l^2}{l^2 + b\pi^2}\frac{2\delta}{q+1}(\frac{\pi}{l})^{q+1}(s^{q+1} - \alpha^{q+1}) - \frac{l^2\pi^2 + a\pi^4}{l^2(l^2 + b\pi^2)}(s^2 - \alpha^2) + \beta^2\Big]^{-\frac{1}{2}}\mathrm{d}s < +\infty$$

and $\varphi(t)$ develops a singularity in finite time $T_0 \leq T_1$. Obviously, because of $\varphi(t) > 0$, there is the fact that

$$\varphi(t) \leq \sup_{x \in (0,\ l)} |u(\cdot, t)|,$$

therefore it follows that

$$\sup_{x \in (0,\ l)} |u(\cdot, t)| \to +\infty$$

as $t \to T_0^-$. Theorem 3.2 is proved. □

4. The problem (1.2), (1.3) and (1.6)

In this section we apply the above conclusion to the problem (1.2), (1.3) and (1.6).

By the contraction mapping principle [15] we can prove that the problem (1.2), (1.3) and (1.6) have a unique local generalized solution and a unique local classical solution. By the aid of Theorem 3.2, we have the following theorem:

Theorem 4.1 *Suppose that $u(x,t)$ is the generalized solution of the problem (1.2), (1.3) and*

(1.6), and the following assumptions hold:

$$\frac{\pi}{2l}\int_0^l u_0(x)\cos\frac{\pi x}{l}\mathrm{d}x=\alpha\geq\frac{\pi^2l^2+a\pi^4}{l^4}(\frac{l}{\pi})^3,\ \frac{\pi}{2l}\int_0^l u_1(x)\cos\frac{\pi x}{l}\mathrm{d}x=\beta>0.$$

Then the solution $u(x,t)$ of the problem (1.2), (1.3) and (1.6) blows up in finite time T_0, i.e.

$$\lim_{t\to T_0^-}\sup_{x\in(0,\ l)}|u(\cdot,t)|\to+\infty.$$

Proof It is easy to know that the integral

$$T_2=\int_\alpha^{+\infty}\Big[\frac{l^2}{l^2+b\pi^2}\frac{2}{3}(\frac{\pi}{l})^3(s^3-\alpha^3)-\frac{l^2\pi^2+a\pi^4}{l^2(l^2+b\pi^2)}(s^2-\alpha^2)+\beta^2\Big]^{-\frac{1}{2}}\mathrm{d}s<+\infty.$$

Making use of Theorem 3.2, we know that there exists an finite time $T_0<T_2$ such that

$$\lim_{t\to T_0^-}\sup_{x\in(0,\ l)}|u(\cdot,t)|\to+\infty.$$

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