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k-Torsionfree Modules with Respect to Cotilting Modules

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Abstract Let Λ and Γ be left and right Noetherian rings and ${}_\Lambda\omega_\Gamma$ a cotilting bimodule. A necessary and sufficient condition for a finitely generated Λ-module to be ω-k-torsionfree is given and the extension closure of T^i_ω is discussed. As applications, we give some results of ${}_\Lambda\omega_\Gamma$ related to $l.\mathrm{id}(\omega) \leq k$.

Keywords cotilting bimodules; ω-k-torsionfree modules.

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1. Introduction

Throughout this paper Λ is a left Noetherian ring and Γ is a right Noetherian ring. Denote by $\mathrm{mod}\Lambda$ (resp. $\mathrm{mod}\Lambda^{(op)}$) the category of finitely generated left (resp. right) Λ-modules.

Cotilting (bi)modules and homologically finite subcategories are very important research objects in representation theory of algebras, which play very important roles in studying the dual of $\mathrm{mod}\Lambda$ and in determining the existence of almost split sequences in subcategories of $\mathrm{mod}\Lambda$. The definition of cotilting (bi)modules on Noether algebra was given in [1] by Huang. And later he proved that the definition coincides with that given by Auslander in the case of Artin algebras.

In [1], Huang proved that: '${}^\perp\omega$ is functorically finite in $\mathrm{mod}\Lambda$ if ${}_\Lambda\omega_\Gamma$ is a cotilting bimodule', and gave equivalent conditions for finitely generated modules to be ω-torsionless or ω-reflexive. In fact, the notion of ω-k-torsionfree modules is a generalization of the notions of ω-torsionless modules and ω-reflexive modules, and we refer to [1] for the details. As the main result of this paper, we will generalize the results in [1] and give an equivalent condition that a finitely Λ-module is ω-k-torsionfree. Furthermore, as applications, the cotilting bimodule ${}_\Lambda\omega_\Gamma$ with $l.\mathrm{id}(\omega) \leq k$ will be revisited and the extension closure of $T^k_\omega(\Lambda)$ will be considered.

2. Definitions and notations

In this section, we will recall some basic definitions and notations which will be used later.

Definition 2.1 *Assume that* $\mathfrak{C} \supset \mathfrak{D}$ *are subcategories of* $\mathrm{mod}\Lambda$ *and* $C \in \mathfrak{C}$, $D \in \mathrm{add}\mathfrak{D}$,

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where add$\mathfrak{D}$ *is the category of* modΛ *consisting of all* Λ*-modules isomorphic to summand of finite sum of modules in* $\mathfrak{D}$*. The morphism* $D \to C$ *is said to be a right* $\mathfrak{D}$*-approximation of* C *if* $\mathrm{Hom}_\Lambda(X, D) \to \mathrm{Hom}_\Lambda(X, C)$ *is epimorphic for all* $X \in \mathrm{add}\mathfrak{D}$*. The subcategory* $\mathfrak{D}$ *is said to be contravariantly finite in* $\mathfrak{C}$ *if every* C *in* $\mathfrak{C}$ *has a right* $\mathfrak{D}$*-approximation. Dually, the morphism* $C \to D$ *is said to be a left* $\mathfrak{D}$*-approximation of* C *if* $\mathrm{Hom}_\Lambda(D, X) \to \mathrm{Hom}_\Lambda(C, X)$ *is epimorphic for all* $X \in \mathrm{add}\mathfrak{D}$*. The subcategory* $\mathfrak{D}$ *is said to be covariantly finite in* $\mathfrak{C}$ *if every* C *in* $\mathfrak{C}$ *has a left* $\mathfrak{D}$*-approximation. The subcategory* $\mathfrak{D}$ *is said to be functorially finite in* $\mathfrak{C}$ *if it is both contravariantly finite and covariantly finite in* $\mathfrak{C}$*. The notion of contravariantly finite subcategories, covariantly finite subcategories and functorially finite subcategories are referred to as homologically finite subcategories.*

For a left Λ-module(resp. right Λ-module) A, use $l.\mathrm{id}_\Lambda(A)$ (resp. $r.\mathrm{id}_\Lambda(A)$) to denote left (resp. right) injective dimension of A.

Definition 2.2 *Let* $\omega \in \mathrm{mod}\Lambda$*. We call* ω *a selforthogonal module if* $\mathrm{Ext}^i_\Lambda(\omega, \omega) = 0$ *for any* $i \geq 1$*. A selforthogonal module* ω *is called a cotilting module if* $l.\mathrm{id}_\Lambda(\omega) < \infty$ *and the natural map* $\Lambda \to \mathrm{End}(\omega_{\mathrm{End}(_\Lambda\omega)})$ *is an isomorphism. Similarly, we can define the notion of cotilting module in* $\mathrm{mod}\Gamma^{(op)}$*. A* (Λ, Γ)*-bimodule* $_\Lambda\omega_\Gamma$ *is called a cotilting bimodule if* $_\Lambda\omega$ *and* ω_Γ *are cotilting modules and the natural maps* $\Gamma^{(op)} \to \mathrm{End}(_\Lambda\omega)$ *and* $\Lambda \to \mathrm{End}(\omega_\Gamma)$ *are isomorphisms.*

For any $A \in \mathrm{mod}\Lambda$ (resp. $\mathrm{mod}\Gamma^{(op)}$), we use $\mathrm{add}_\Lambda A$ (resp. $\mathrm{add}A_\Gamma$) to denote the full subcategory of modΛ (resp. $\mathrm{mod}\Gamma^{(op)}$) consisting of all modules isomorphic to direct summands of finite direct sums of copies of $_\Lambda A$ (resp. A_Γ). Suppose that $_\Lambda\omega_\Gamma$ is a (Λ, Γ)-bimodule, we put $(-)^\omega = \mathrm{Hom}(-, \omega)$. Let $\sigma_A : A \to A^{\omega\omega}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^\omega$ be the canonical evaluation homomorphism. If σ_A is a monomorphism, then A is called an ω-torsionless module. If σ_A is an isomorphism, then A is called an ω-reflexive module.

Definition 2.3 *Let* $A \in \mathrm{mod}\Lambda$ *and* $P_1 \to P_0 \xrightarrow{f} A \to 0$ *be a projective resolution in* modΛ*. Then we have the following exact sequence:*

$$0 \to A^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to \mathrm{Coker}f^\omega \to 0.$$

Set $\mathrm{Coker}f^\omega = Tr_\omega A$*.* A *is called* ω*-*k*-torsionfree if* $\mathrm{Ext}^i_\Gamma(Tr_\omega A, \omega) = 0$ *for any* $1 \leq i \leq k$*. Denote by* $T^k_\omega(\Lambda)$ *the full subcategory of* modΛ *consisting of all* ω*-*k*-torsionfree modules.*

Remark If Λ is a two sided Noetherian ring and $_\Lambda\omega_\Gamma = _\Lambda \Lambda_\Lambda$, then the notion of Definition 2.3 is just k-torsionfree modules, and we use $T^k(\Lambda)$ to denote the full subcategory of modΛ consisting of all k-torsionfree modules. The rationality of the Definition 2.3 is proved in [3].

Let $\omega \in \mathrm{mod}\Lambda$ be a selforthogonal module and $X \in \mathrm{mod}\Lambda$. X is said to be left orthogonal with ω if $\mathrm{Ext}^i_\Lambda(X, \omega) = 0$ for any $i \geq 1$. We use $^\perp\omega$ to denote the subcategory of modΛ consisting of the modules which are left orthogonal with ω.

3. The main results

In the following, we assume that ω is a (Λ, Γ)-cotilting bimodule.

Lemma 3.1 (Lemma 2.1 in [4]) *Let $A \in \mathrm{mod}\Lambda$. Then we have the following exact sequence:*

$$0 \to \mathrm{Ext}^1_\Gamma(Tr_\omega A, \omega) \to A \xrightarrow{\sigma_A} A^{\omega\omega} \to \mathrm{Ext}^2_\Gamma(Tr_\omega A, \omega) \to 0.$$

By Definition 2.3, we have

Corollary 3.2 *Let $A \in \mathrm{mod}\Lambda$. Then A is an ω-torsionless module if and only if A is an ω-1-torsionfree module, and A is an ω-reflexive module if and only if A is an ω-2-torsionfree module.*

Remark For any $A \in {}^\perp \omega$ (where $A \in \mathrm{mod}\Lambda$), by the proof of Theorem 6.1 in [5], $A^\omega \in \mathrm{mod}\Gamma^{(op)}$ and $A^\omega \in {}^\perp \omega$. Let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective resolution of A in $\mathrm{mod}\Lambda$. Then we have the following exact sequence:

$$0 \to A^\omega \to P_0^\omega \xrightarrow{f^\omega} P_1^\omega \to Tr_\omega A \to 0$$

such that for any $i \geq 1$, $\mathrm{Ext}^i_\Gamma(Tr_\omega A, \omega) \cong \mathrm{Ext}^{i+2}_\Gamma(A^\omega, \omega) = 0$. By Definition 2.3, A is an ω-k-torsionfree module for any $k \geq 1$.

Suppose that $\mathfrak{D}$ is a subcategory of $\mathrm{mod}\Lambda$. It is straightforward from Definition 2.1 to verify that if one of the right $\mathfrak{D}$-approximations of a module in $\mathrm{mod}\Lambda$ is epimorphic, then all of the right $\mathfrak{D}$-approximations of this module are epimorphic. Dually, if one of the left $\mathfrak{D}$-approximations of a module in $\mathrm{mod}\Lambda$ is monomorphic, then all of the left $\mathfrak{D}$-approximations of this module are monomorphic.

Lemma 3.3 (Theorem 1 in [1]) *${}^\perp\omega$ is functorially finite in* $\mathrm{mod}\Lambda$.

In [1], Huang classified the modules in $\mathrm{mod}\Lambda$ by using the properties of monomorphic left ${}^\perp\omega$-approximation and got the following result:

Theorem 3.4 *Let $C \in \mathrm{mod}\Lambda$. Then*

(1) C is ω-torsionless if and only if there is an exact sequence $0 \to C \xrightarrow{f} X$ such that $f: C \to X$ is a left ${}^\perp\omega$-approximation of C (i.e., C has a monomorphic left ${}^\perp\omega$-approximation).

(2) C is ω-reflexive if and only if there is an exact sequence $0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$ such that $f_1 : C \to X_1$ and $\mathrm{Im} f_2 \to X_2$ are left ${}^\perp\omega$-approximation of C and $\mathrm{Im} f_2$, respectively.

We know that the notion of ω-k-torsionfree modules is generalizations of ω-torsionless modules and ω-reflexive modules. In fact, we can generalize the theorem above to the case for ω-k-torsionfree modules. The following is the main result of this paper.

Theorem 3.5 *Let $C \in \mathrm{mod}\Lambda$. Then C is an ω-k-torsionfree module if and only if there is an exact sequence*

$$0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_k} X_k$$

such that $\mathrm{Im} f_i \to X_i$ is a left ${}^\perp\omega$-approximation of $\mathrm{Im} f_i$ for each $1 \leq i \leq k$.

Proof Proceed by induction on k. It is not difficult to verify the case for $k = 1, 2$ by Corollary 3.2 and Theorem 3.4. Now suppose $k \geq 3$.

($\Rightarrow$) Suppose that C is an ω-k-torsionfree module. Then it is ω-torsionless and ω-reflexive clearly. By Corollary 3.2 and Theorem 3.4, there is an exact sequence $0 \to C \xrightarrow{f_1} X_1 \to N \to 0$

such that $f_1 : 0 \to C \to X_1$ is a monomorphic left ${}^{\perp}\omega$-approximation of C. We have an exact sequence $0 \to N^{\omega} \to X_1^{\omega} \xrightarrow{f_1^{\omega}} C^{\omega} \to 0$ by Definition 2.1.

By the remark above, $X_1^{\omega} \in \mathrm{mod}\Gamma^{(op)}$ and $X_1^{\omega} \in^{\perp} \omega$ if $X_1 \in^{\perp} \omega$. So we have $\mathrm{Ext}_{\Gamma}^{i}(N^{\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i+1}(C^{\omega}, \omega)$ for any $i \geq 1$. Since C is an ω-k-torsionfree module, $\mathrm{Ext}_{\Gamma}^{i}(Tr_{\omega}C, \omega) = 0$ for any $1 \leq i \leq k$ and hence $Ext_{\Gamma}^{i}(C^{\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i+2}(Tr_{\omega}C, \omega) = 0$ for any $1 \leq i \leq k-2$, which results in

$$\mathrm{Ext}_{\Gamma}^{i+2}(Tr_{\omega}N, \omega) \cong \mathrm{Ext}_{\Gamma}^{i}(N^{\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i+1}(Tr_{\omega}C, \omega) = 0, \ \forall 1 \leq i \leq k-3.$$

Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \xrightarrow{f} & X_1 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_C} & & \downarrow{\scriptstyle \sigma_{X_1}} & & \downarrow{\scriptstyle \sigma_N} & & \\
0 & \longrightarrow & C^{\omega\omega} & \longrightarrow & X_1^{\omega\omega} & \longrightarrow & N^{\omega\omega} & \longrightarrow & 0
\end{array}$$

where σ_C, σ_{X_1} are isomorphism. Obviously, σ_N is an isomorphism. By Lemma 3.1, we have

$$\mathrm{Ext}_{\Gamma}^{1}(Tr_{\omega}N, \omega) = 0 = \mathrm{Ext}_{\Gamma}^{2}(Tr_{\omega}N, \omega).$$

So $\mathrm{Ext}_{\Gamma}^{i}(Tr_{\omega}N, \omega) = 0$ for any $1 \leq i \leq k-1$ and hence N is an ω-$(k-1)$-torsionfree module.

($\Leftarrow$) Suppose that there is an exact sequence

$$0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_k} X_k$$

such that $\mathrm{Im}f_i \to X_i$ is a left ${}^{\perp}\omega$-approximation of $\mathrm{Im}f_i$ for each i. Put $N = \mathrm{Im}f_2$. Then we have the following exact sequence:

$$0 \to C \xrightarrow{f_1} X_1 \xrightarrow{f_2} N \to 0,$$

where N is an ω-$(k-1)$-torsionfree module. Thus N is ω-torsionless and $\mathrm{Ext}_{\Gamma}^{i}(N^{\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i+2}(Tr_{\omega}N, \omega) = 0$ for any $1 \leq i \leq k-3$.

Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & X_1 & \xrightarrow{f_2} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_C} & & \downarrow{\scriptstyle \sigma_{X_1}} & & \downarrow{\scriptstyle \sigma_N} & & \\
0 & \longrightarrow & C^{\omega\omega} & \longrightarrow & X_1^{\omega\omega} & \xrightarrow{f_2^{\omega\omega}} & N^{\omega\omega} & &
\end{array}$$

Since σ_N and σ_{X_1} are isomorphism, which implies that σ_C is an isomorphism and $f_2^{\omega\omega}$ is epimorphic, we have C is an ω-torsionless module and $\mathrm{Ext}_{\Gamma}^{1}(C^{\omega}, \omega) = 0$. We have the exact sequence $0 \to N^{\omega} \to X_1^{\omega} \to C^{\omega} \to 0$ by the definition of approximation. And we have $\mathrm{Ext}_{\Gamma}^{i}(C^{\omega}, \omega) \cong \mathrm{Ext}_{\Gamma}^{i-1}(N^{\omega}, \omega) = 0$ for any $2 \leq i \leq k-2$. Hence $\mathrm{Ext}_{\Gamma}^{i}(C^{\omega}, \omega) = 0$ for any $1 \leq i \leq k-2$.

Let $P_1 \to P_0 \to C \to 0$ be a projective resolution of C in $\mathrm{mod}\Lambda$. We have the exact sequence

$$0 \to C^{\omega} \to P_0^{\omega} \to P_1^{\omega} \to Tr_{\omega}C \to 0,$$

where P_0, P_1 are projective left Λ-modules. So $\mathrm{Ext}_{\Gamma}^{i}(Tr_{\omega}C, \omega) \cong \mathrm{Ext}_{\Gamma}^{i-2}(C^{\omega}, \omega) = 0$ for any $3 \leq i \leq k$. Note that C is also ω-torsionless and ω-reflexive, and C is an ω-k-torsionfree module.

In the following, we will deal with the extension closure of $T_\omega^k(\Lambda)$ when ${}_\Lambda\omega_\Gamma$ is a cotilting bimodule.

Definition 3.6 *A full subcategory χ of* $\mathrm{mod}\Lambda$ *is said to be extension closed if the middle term B of any short exact sequence $0 \to A \to B \to C \to 0$ is in χ provided that A and C are in χ.*

For extension closure of $T_\omega^i(\Lambda)$, we have:

Theorem 3.7 *If ${}_\Lambda\omega_\Gamma$ is a cotilting bimodule, then $T_\omega^i(\Lambda)$ is extension closed for any $i \geq 1$.*

Proof For $i = 1$, let $0 \to A \to B \to C \to 0$ be a short exact sequence with $A,\ C \in T_\omega^1(\Lambda)$. We claim that $B \in T_\omega^1(\Lambda)$.

In fact, $A,\ C \in T_\omega^1(\Lambda)$ if and only if there are exact sequences $0 \to A \xrightarrow{f_A} X_A$, $0 \to C \xrightarrow{f_C} X_C$ such that $f_A : A \to X_A$, $f_C : C \to X_C$ are monomorphic left ${}^\perp\omega$-approximation of A and C by Theorem 3.4, respectively. Hence we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \\
0 & \longrightarrow & X_A & \longrightarrow & X_A \bigoplus X_C & \longrightarrow & X_C & \longrightarrow & 0
\end{array}$$

where $X_A, X_C \in {}^\perp\omega$, $X_A \bigoplus X_C \in {}^\perp\omega$. Put $\mathrm{Hom}(-, D) = (-)^D$ for any $D \in \mathrm{add}\,{}^\perp\omega$. Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_C^D & \longrightarrow & X_C^D \bigoplus X_A^D & \longrightarrow & X_A^D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_C^D} & & \downarrow{\scriptstyle f_B^D} & & \downarrow{\scriptstyle f_A^D} & & \\
0 & \longrightarrow & C^D & \longrightarrow & B^D & \longrightarrow & A^D & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathrm{Coker} f_B^D & & 0 & &
\end{array}$$

By 'Snake Lemma', $\mathrm{Coker} f_B^D = 0$. Hence $0 \to B \xrightarrow{f_B} X_A \bigoplus X_C$ is a monomorphic left ${}^\perp\omega$-approximation of B. By Theorem 3.4, B is a ω-torsionless module, i.e., $B \in T_\omega^1(\Lambda)$.

For $i = 2$, similarly, by Theorem 3.4, there exist exact sequences

$$0 \to A \xrightarrow{f_{1A}} X_{1A} \xrightarrow{f_{2A}} X_{2A},$$

$$0 \to C \xrightarrow{f_{1C}} X_{1C} \xrightarrow{f_{2C}} X_{2C}$$

such that $f_{1A} : A \to X_{1A}$ and $\mathrm{Im} f_{2A} \to X_{2A}$ are left ${}^\perp\omega$-approximation of A and $\mathrm{Im} f_{2A}$, respectively. At the same time, $f_{1C} : C \to X_{1C}$ and $\mathrm{Im} f_{2C} \to X_{2C}$ are left ${}^\perp\omega$-approximation of C and $\mathrm{Im} f_{2C}$, respectively.

Similarly to the proof of the case $i = 1$, we get $0 \to B \xrightarrow{f_{1B}} X_{1A} \bigoplus X_{1C}$ is a left ${}^\perp\omega$-

approximation of B. Consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Im} f_{1A} & \longrightarrow & \mathrm{Im} f_{1A} \bigoplus \mathrm{Im} f_{1C} & \longrightarrow & \mathrm{Im} f_{1C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{2A} & \longrightarrow & X_{2A} \bigoplus X_{2C} & \longrightarrow & X_{2C} & \longrightarrow & 0
\end{array}$$

By the process of proof of the case for $i = 1$, $\mathrm{Im} f_{1A} \bigoplus \mathrm{Im} f_{1C} \xrightarrow{f_{2B}} X_{2A} \bigoplus X_{2C}$ is a left ${}^{\perp}\omega$-approximation of $\mathrm{Im} f_{1A} \bigoplus \mathrm{Im} f_{1C}$.

Therefore, we have an exact sequence:

$$0 \to B \xrightarrow{f_{1B}} X_{1A} \bigoplus X_{1C} \xrightarrow{f_{2B}} X_{2A} \bigoplus X_{2C}$$

such that $f_{1B} : B \to X_{1A} \bigoplus X_{1C}$, $\mathrm{Im} f_{1A} \bigoplus \mathrm{Im} f_{1C} \cong (\mathrm{Im} f_{2B}) \to X_{2A} \bigoplus X_{2C}$ are left ${}^{\perp}\omega$-approximation of B and $\mathrm{Im} f_{2B}$, respectively. It follows from Theorem 3.4 that B is an ω-reflexive module, i.e., $B \in T_{\omega}^{2}(\Lambda)$.

Our conclusion follows from repeating the process of the above proof and Theorem 3.5.

In the following, we discuss the case for $l.\mathrm{id}(\omega) \leq k$.

Definition 3.8 *Let* $A \in \mathrm{mod}\Lambda$ *and* k *be a positive integer. We call* A *an* ω-k-*syzygy module if there is an exact sequence* $0 \to A \to X_0 \to X_1 \to \cdots \xrightarrow{f_{k-1}} X_{k-1}$ *with all* $X_i \in \mathrm{add}_{\Lambda}\omega$ *for any* $0 \leq i \leq k-1$. *Where* $\mathrm{add}_{\Lambda}\omega$ *denotes the full subcategory of* $\mathrm{mod}\Lambda$ *consisting of all modules isomorphic to the direct summands of finite direct sums of copies of* ${}_{\Lambda}\omega$. *Furthermore, we call* $\mathrm{Coker} f_{k-1}$ *an* ω-k-*cosyzygy module. We use* $\Omega^{k}(\Lambda)$ *and* $\Omega^{-k}(\Lambda)$ *to denote the full subcategory of* $\mathrm{mod}\Lambda$ *consisting of all* ω-k-*syzygy modules and* ω-k-*cosyzygy modules, respectively.*

Remark If Λ is a two sided noetherian ring and ${}_{\Lambda}\omega_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$, then Definition 3.8 is just k-zysygy modules. We use $\Omega^{k}(\Lambda)$ to denote the full subcategory of $\mathrm{mod}\Lambda$ consisting of all k-syzygy modules. In [3], Huang proved that every ω-k-torsionfree module is an ω-k-syzygy module, i.e., $T_{\omega}^{k}(\Lambda) \subseteq \Omega_{\omega}^{k}(\Lambda)$. When $T_{\omega}^{k}(\Lambda) = \Omega_{\omega}^{k}(\Lambda)$? We will give a sufficient condition for this question.

Lemma 3.9 *If* $l.\mathrm{id}(\omega) \leq k$, *then* ${}^{\perp}\omega = \Omega_{\omega}^{k}(\Lambda)$.

Proof Suppose that $C \in \Omega_{\omega}^{k}$. By Definition 3.8, there is an exact sequence

$$0 \to C \to X_0 \to X_1 \to \cdots \xrightarrow{f_{k-1}} X_{k-1} \to \mathrm{Coker} f_{k-1} \to 0$$

with all $X_i \in \mathrm{add}_{\Lambda}\omega$. Since $l.\mathrm{id}(\omega) \leq k$, $\mathrm{Ext}_{\Lambda}^{i}(C, \omega) \cong \mathrm{Ext}_{\Lambda}^{i+k}(\mathrm{Coker} f_{k-1}, \omega) = 0$ for any $i \geq 1$ and hence $C \in^{\perp} \omega$.

On the other hand, let $C \in^{\perp} \omega$ and

$$\cdots \to P_t \to \cdots \to P_1 \to P_0 \to C^{\omega} \to 0$$

be a projective resolution of C^{ω} in $\mathrm{mod}\Gamma^{(op)}$. By Theorem 6.1 in [5], $\mathrm{Ext}_{\Gamma}^{i}(C^{\omega}, \omega) = 0$ for any

$i \geq 1$. And $C \cong C^{\omega\omega}$ (i.e., C is an ω-reflexive module) by aforementioned remark. Hence we have the following exact sequence:

$$0 \to C \cong C^{\omega\omega} \to P_0^{\omega} \to P_1^{\omega} \to \cdots \to P_t^{\omega} \to \cdots,$$

where all $P_i^{\omega} \in \mathrm{add}_{\Lambda}\omega$. Hence $C \in \Omega_{\omega}^{k}$.

The following corollary is an immediate consequence of Lemmas 3.3 and 3.9.

Corollary 3.10 *If* ${}_{\Lambda}\omega_{\Gamma}$ *is a cotilting bimodule and* $l.\mathrm{id}(\omega) \leq k$, *then* $\Omega_{\omega}^{k}(\Lambda)$ *is functorially finite in* $\mathrm{mod}\Lambda$.

Corollary 3.11 *If* ${}_{\Lambda}\omega_{\Gamma}$ *is a cotilting bimodule and* $l.\mathrm{id}(\omega) \leq k$, *then* $T_{\omega}^{k}(\Lambda) = \Omega_{\omega}^{k}(\Lambda)$.

Proof First, the author proved that $T_{\omega}^{k} \subseteq \Omega_{\omega}^{k}(\Lambda)$ in [3]. On the other hand, we have ${}^{\perp}\omega = \Omega_{\omega}^{k}(\Lambda)$ by Lemma 3.9. For any $C \in \Omega_{\omega}^{k}(\Lambda)$, there is an exact sequence $0 \to C \to C \to 0 \to \cdots \to 0$ which satisfies the condition in Theorem 3.5. Hence $C \in T_{\omega}^{k}(\Lambda)$.

Let ${}_{\Lambda}\omega_{\Gamma} = {}_{\Lambda}\Lambda_{\Lambda}$. We have:

Corollary 3.12 *If* ${}_{\Lambda}\Lambda_{\Lambda}$ *is a cotilting bimodule and* $l.\mathrm{id}(\Lambda) \leq k$, *then* $T^{k}(\Lambda) = \Omega^{k}(\Lambda)$.

For extension closure of $T_{\omega}^{k}(\Lambda)$, we have

Theorem 3.13 *If* ${}^{\perp}\omega = \Omega_{\omega}^{k}(\Lambda)$, *then* $T_{\omega}^{k}(\Lambda)$ *is extension closed.*

Proof Suppose that $0 \to A \xrightarrow{f} B \to C \to 0$ is a short exact sequence in $\mathrm{mod}\Lambda$ with $A, C \in T_{\omega}^{k}(\Lambda)$. We claim that $B \in T_{\omega}^{k}(\Lambda)$. Consider the following commutative diagram with first two rows splitting:

$$\begin{array}{ccccccccc}
0 & \to & F_1 & \to & F_1 \bigoplus G_1 & \to & G_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_0 & \to & F_0 \bigoplus G_0 & \to & G_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \xrightarrow{f} & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all P_i and G_i are projective. Then we get the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{\omega} & \to & B^{\omega} & \xrightarrow{f^{\omega}} & A^{\omega} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_0^{\omega} & \to & G_0^{\omega} \bigoplus F_0^{\omega} & \to & F_0^{\omega} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_1^{\omega} & \to & G_1^{\omega} \bigoplus F_1^{\omega} & \to & F_1^{\omega} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Z & & Y & & X & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which leads to the following exact sequence:

$$0 \to C^{\omega} \to B^{\omega} \xrightarrow{f^{\omega}} A^{\omega} \to Z \to Y \to X \to 0.$$

Because $C \in T_{\omega}^{k}(\Lambda)$, $C \in \Omega^{k}(\Lambda)$ and $C \in^{\perp} \omega$ by assumption. We have $\mathrm{Ext}_{\Lambda}^{1}(C,\omega) = 0$ and f^{ω} is epimorphic. So there is an exact sequence $0 \to Z \to Y \to X \to 0$. Since $A, C \in T_{\omega}^{k}$, $\mathrm{Ext}_{\Gamma}^{i}(X,\omega) = 0 = \mathrm{Ext}_{\Gamma}^{i}(Z,\omega)$ for any $1 \leq i \leq k$. So we have $\mathrm{Ext}_{\Gamma}^{i}(Y,\omega) = 0$ for any $1 \leq i \leq k$. By the long exact sequence theorem, we have B is an ω-k-torsionfree module.

The following conclusion is immediate from Lemma 3.9 and Theorem 3.13.

Corollary 3.14 (Corollary 4.1 in [6]) *If* ${}_{\Lambda}\omega_{\Gamma}$ *is a cotilting bimodule and* $l.\mathrm{id}(\omega) \leq k$, *then* $T_{\omega}^{k}(\Lambda)$ *is extension closed.*

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