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A Class of Maximal General Armendariz Subrings of Matrix Rings

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Abstract An associative ring with identity R is called Armendariz if, whenever $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0$ in $R[x]$, $a_i b_j = 0$ for all i and j. An associative ring with identity is called reduced if it has no non-zero nilpotent elements. In this paper, we define a general reduced ring (with or without identity) and a general Armendariz ring (with or without identity), and identify a class of maximal general Armendariz subrings of matrix rings over general reduced rings.

Keywords general Armendariz ring; matrix ring; general reduced ring.

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1. Introduction

According to Rege and Chhawchharia[7], a ring R is called Armendariz if, whenever

$$(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0$$

in $R[x]$, $a_i b_j = 0$ for all i and j. A ring is called reduced if it has no non-zero nilpotent elements. Every reduced ring is Armendariz by Armendariz[2], but the more comprehensive study of the notion of Armendariz rings was carried out just recently[1,3−6,8,9].

Rege and Chhawchharia[7] showed that every n-by-n full matrix ring over any ring is not Armendariz, where $n \geq 2$. For a reduced ring R, it is interesting to find some general Armendariz subrings of matrix rings. In this paper, a class of maximal general Armendariz subrings of matrix rings are described.

By the term "ring" we mean an associative ring with identity, and by a general ring we mean an associative ring with or without identity. For clarity, R will always denote a ring, and a general ring will be denoted by I. We write $M_n(R)$ for the n-by-n full matrix ring over a ring R.

2. Main results

Definition 2.1 *A general ring I is called general reduced if it has no non-zero nilpotent elements.*

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Definition 2.2 *A general ring I is called general Armendariz if, whenever*

$$(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) = 0$$

in $I[x]$, $a_i b_j = 0$ for all i and j.

Clearly any general reduced ring is general Armendariz. In the following we will see the converse is not true.

Let $D_n(I) = \left\{ \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & a_n \end{pmatrix} | a_i \in I \right\}$ for $n \geq 2$.

Lemma 2.3 *Let I be a general reduced. If $ab = 0$ for $a, b \in I$, then $ba = 0$.*

Proof Since $ab = 0$ for $a, b \in I$, $(ba)^2 = 0$. Thus $ba = 0$ because I is a general reduced ring.

Theorem 2.4 *If I is a general reduced ring, then $D_n(I)$ is a general Armendariz subring of $M_n(I)$ for $n \geq 2$.*

Proof Suppose that $f(x) = \sum_{i=0}^{m} A_i x^i, g(x) = \sum_{j=0}^{m} B_j x^j \in D_n(I)[x]$, such that $f(x)g(x) = 0$. We need to prove that $A_i B_j = 0$ for all i and j. Let

$$A_i = \begin{pmatrix} a_1^{(i)} & a_1^{(i)} & \cdots & a_1^{(i)} \\ a_2^{(i)} & a_2^{(i)} & \cdots & a_2^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(i)} & a_n^{(i)} & \cdots & a_n^{(i)} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_1^{(j)} & b_1^{(j)} & \cdots & b_1^{(j)} \\ b_2^{(j)} & b_2^{(j)} & \cdots & b_2^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ b_n^{(j)} & b_n^{(j)} & \cdots & b_n^{(j)} \end{pmatrix},$$

where $a_s^{(i)}, b_s^{(j)} \in I$ for $0 \leq i, j \leq m, 1 \leq s \leq n$.

It follows from $f(x)g(x) = 0$ that

$$\sum_{i+j=l} A_i B_j = 0 \quad \text{for} \quad 0 \leq l \leq 2m. \tag{2.1}$$

We will show that $A_i B_j = 0$ by induction on $i + j$.

If $i + j = 0$, then $A_0 B_0 = 0$ by (2.1).

Now suppose that there exists a positive integer k such that $A_i B_j = 0$ when $i + j < k$. It follows from $A_i B_j = 0$ when $i + j < k$ that

$$a_s^{(i)}[b_1^{(j)} + b_2^{(j)} + \cdots + b_n^{(j)}] = 0 \quad \text{for} \quad i + j < k \quad \text{and} \quad 1 \leq s \leq n. \tag{2.2}$$

By Lemma 2.3, we have

$$[b_1^{(j)} + b_2^{(j)} + \cdots + b_n^{(j)}]a_s^{(i)} = 0 \quad \text{for} \quad i + j < k \quad \text{and} \quad 1 \leq s \leq n. \tag{2.3}$$

We will show that $A_i B_j = 0$ when $i + j = k$. From (2.1), we get

$$A_0 B_k + A_1 B_{k-1} + \cdots + A_k B_0 = 0. \tag{2.4}$$

That is,

$$a_i^{(0)}(\sum_{s=1}^{n} b_s^{(k)}) + a_i^{(1)}(\sum_{s=1}^{n} b_s^{(k-1)}) + \cdots + a_i^{(k)}(\sum_{s=1}^{n} b_s^{(0)}) = 0 \quad \text{for} \quad 1 \le i \le n. \tag{2.5}$$

Thus, multiplying (2.5) by the $(a_i^{(s)})'$s from the right by (2.3) leads to

$$a_i^{(s)}[b_1^{(k-s)} + b_2^{(k-s)} + \cdots + b_n^{(k-s)}] = 0 \quad \text{for} \quad 0 \le s \le k \quad \text{and} \quad 1 \le i \le n. \tag{2.6}$$

Hence we show that $A_iB_j = 0$ when $i + j = k$ by (2.6). Therefore, by induction, $A_iB_j = 0$ for any i and j.

Example 2.5 Let R be a reduced ring. Then $D_2(R)$ is a general Armendariz subring of $M_2(R)$ by Theorem 2.4. Since

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2 = 0,$$

but $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq 0$. Hence $D_2(R)$ is not general reduced.

Theorem 2.6 *If I is a general reduced ring, then $D_n(I)$ is a maximal general Armendariz subring of $M_n(I)$ for $n \ge 2$.*

Proof Suppose that T is a general Armendariz subring of $M_n(I)$ and T properly contains $D_n(I)$. Then there exists $A = (a_{i,j}) \in T \backslash D_n(I)$ where $1 \le i, j \le n$. It suffices to show that T is not general Armendariz. We will proceed with the following two cases.

Case 1 Suppose that $a_{11} = a_{12} = \cdots = a_{1,j-1} \neq a_{1,j}$ where $2 \le j \le n$. Then $a_{1,j-1} - a_{1,j} \neq 0$. Let

$$A_1 = A - \begin{pmatrix} a_{1,j} & a_{1,j} & \cdots & a_{1,j} \\ a_{2,j} & a_{2,j} & \cdots & a_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j} & a_{n,j} & \cdots & a_{n,j} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} & 0 & a_{1,j+1} - a_{1,j} & \cdots & a_{1,n} - a_{1,j} \\ a_{21} - a_{2,j} & \cdots & a_{2,j-1} - a_{2,j} & 0 & a_{2,j+1} - a_{2,j} & \cdots & a_{2,n} - a_{2,j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j} & \cdots & a_{n,j-1} - a_{n,j} & 0 & a_{n,j+1} - a_{n,j} & \cdots & a_{n,n} - a_{n,j} \end{pmatrix},$$

$$A_2 = A - \begin{pmatrix} a_{1,j-1} & a_{1,j-1} & \cdots & a_{1,j-1} \\ a_{2,j-1} & a_{2,j-1} & \cdots & a_{2,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j-1} & a_{n,j-1} & \cdots & a_{n,j-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - a_{1,j-1} & \cdots & a_{1,j-2} - a_{1,j-1} & 0 & a_{1,j} - a_{1,j-1} & \cdots & a_{1,n} - a_{1,j-1} \\ a_{21} - a_{2,j-1} & \cdots & a_{2,j-2} - a_{2,j-1} & 0 & a_{2,j} - a_{2,j-1} & \cdots & a_{2,n} - a_{2,j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j-1} & \cdots & a_{n,j-2} - a_{n,j-1} & 0 & a_{n,j} - a_{n,j-1} & \cdots & a_{n,n} - a_{n,j-1} \end{pmatrix}.$$

Then $A_1, A_2 \in T$.

Let $f(x) = A_1 + A_2x$ be in $T[x]$. Let

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \ (j),$$

$$B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{1,j-1} - a_{1,j} & a_{1,j-1} - a_{1,j} & \cdots & a_{1,j-1} - a_{1,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \ (j-1). \qquad \text{Then } B_1, B_2 \in T.$$

Let $g(x) = B_1 + B_2 x$ be in $T[x]$. Then $f(x)g(x) = 0$, but

$$A_1 B_2 = \begin{pmatrix} (a_{1,j-1} - a_{1,j})^2 & \cdots & (a_{1,j-1} - a_{1,j})^2 \\ (a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{2,j-1} - a_{2,j})(a_{1,j-1} - a_{1,j}) \\ \vdots & \ddots & \vdots \\ (a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j}) & \cdots & (a_{n,j-1} - a_{n,j})(a_{1,j-1} - a_{1,j}) \end{pmatrix} \neq 0.$$

This is a contradiction.

Case 2 Suppose that $a_{t,1} = a_{t,2} = \cdots = a_{t,n}$ where $1 \le t \le i-1$, and $a_{i,1} = a_{i,2} = \cdots = a_{i,j-1} \neq a_{i,j}$, where $1 < i, j \le n$. Then $a_{i,j-1} - a_{i,j} \neq 0$. Let

$$A_1 = A - \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,i-1} & a_{i-1,i-1} & \cdots & a_{i-1,i-1} \\ a_{i,j} & a_{i,j} & \cdots & a_{i,j} \\ a_{i+1,j} & a_{i+1,j} & \cdots & a_{i+1,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j} & a_{n,j} & \cdots & a_{n,j} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{i,1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} & 0 & a_{i,j+1} - a_{i,j} & \cdots & a_{i,n} - a_{i,j} \\ a_{i+1,1} - a_{i+1,j} & \cdots & a_{i+1,j-1} - a_{i+1,j} & 0 & a_{i+1,j+1} - a_{i+1,j} & \cdots & a_{i+1,n} - a_{i+1,j} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j} & \cdots & a_{n,j-1} - a_{n,j} & 0 & a_{n,j+1} - a_{n,j} & \cdots & a_{n,n} - a_{n,j} \end{pmatrix},$$

$$A_2 = A - \begin{pmatrix} a_{11} & a_{11} & \cdots & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,i-1} & a_{i-1,i-1} & \cdots & a_{i-1,i-1} \\ a_{i,j-1} & a_{i,j-1} & \cdots & a_{i,j-1} \\ a_{i+1,j-1} & a_{i+1,j-1} & \cdots & a_{i+1,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,j-1} & a_{n,j-1} & \cdots & a_{n,j-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & & & & & & \\ a_{i,1} - a_{i,j-1} & \cdots & a_{i,j-2} - a_{i,j-1} & 0 & a_{i,j} - a_{i,j-1} & \cdots & a_{i,n} - a_{i,j-1} \\ a_{i+1,1} - a_{i+1,j-1} & \cdots & a_{i+1,j-2} - a_{i+1,j-1} & 0 & a_{i+1,j} - a_{i+1,j-1} & \cdots & a_{i+1,n} - a_{i+1,j-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - a_{n,j-1} & \cdots & a_{n,j-2} - a_{n,j-1} & 0 & a_{n,j} - a_{n,j-1} & \cdots & a_{n,n} - a_{n,j-1} \end{pmatrix}.$$

Then $A_1, A_2 \in T$.

Let $f(x) = A_1 + A_2 x$ be in $T[x]$. Let

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{i,j-1} - a_{i,j} & a_{i,j-1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (j),$$

$$B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_{i,j-1} - a_{i,j} & a_{i,j-1} - a_{i,j} & \cdots & a_{i,j-1} - a_{i,j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (j-1). \qquad \text{Then } B_1, B_2 \in T.$$

Let $g(x) = B_1 + B_2 x$ be in $T[x]$. Then $f(x)g(x) = 0$, but

$$A_1 B_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ (a_{i,j-1} - a_{i,j})^2 & \cdots & (a_{i,j-1} - a_{i,j})^2 \\ (a_{i+1,j-1} - a_{i+1,j})(a_{i,j-1} - a_{i,j}) & \cdots & (a_{i+1,j-1} - a_{i+1,j})(a_{i,j-1} - a_{i,j}) \\ \vdots & \ddots & \vdots \\ (a_{n,j-1} - a_{n,j})(a_{i,j-1} - a_{i,j}) & \cdots & (a_{n,j-1} - a_{n,j})(a_{i,j-1} - a_{i,j}) \end{pmatrix} \neq 0.$$

This is a contradiction. Thus T is not general Armendariz.

References

[1] ANDERSON D D, CAMILLO V. *Armendariz rings and Gaussian rings* [J]. Comm. Algebra, 1998, **26**(7): 2265–2272.
[2] ARMENDARIZ E P. *A note on extensions of Baer and P.P.-rings* [J]. J. Austral. Math. Soc., 1974, **18**: 470–473.
[3] HONG C Y, KIM N K, KWAK T K. *Ore extensions of Baer and p.p.-rings* [J]. J. Pure Appl. Algebra, 2000, **151**(3): 215–226.
[4] HUH C, LEE Y, SMOKTUNOWICZ A. *Armendariz rings and semicommutative rings* [J]. Comm. Algebra, 2002, **30**(2): 751–761.
[5] KIM N K, LEE Y. *Armendariz rings and reduced rings* [J]. J. Algebra, 2000, **223**(2): 477–488.
[6] LEE T K, WONG T L. *On Armendariz rings* [J]. Houston J. Math., 2003, **29**(3): 583–593.
[7] REGE M B, CHHAWCHHARIA S. *Armendariz rings* [J]. Proc. Japan Acad. Ser. A Math. Sci., 1997, **73**(1): 14–17.
[8] LEE T K, ZHOU Yiqiang. *Armendariz and reduced rings* [J]. Comm. Algebra, 2004, **32**(6): 2287–2299.
[9] LIU Zhongkui. *Armendariz rings relative to a monoid* [J]. Comm. Algebra, 2005, **33**(3): 649–661.