Generalized Semi-$\pi$-Regular Rings

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Abstract In this paper, the concept of right generalized semi-$\pi$-regular rings is defined. We prove that these rings are non-trivial generalizations of both right GP-injective rings and semi-$\pi$-regular rings. Some properties of these rings are studied and some results about generalized semiregular rings and GP-injective rings are extended.

Keywords GP-injective rings; semi-$\pi$-regular rings; generalized semiregular rings; generalized semi-$\pi$-regular rings.

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1. Introduction

Throughout this paper, the ring $R$ is always associative with identity and all modules are unitary. For any non-empty subset $X$ of $R$, the right (resp. left) annihilator of $X$ in $R$ will be denoted by $r(X)$ (resp. $l(X)$). The symbols $J(R)$, $Z(RR)$, $Z(RR)$ will denote the Jacobson radical, the right singular ideal and the left singular ideal of $R$, respectively. See [1–3] for the other undefined concepts and notations.

A module $M$ is said to be AP-injective$^4$, if for any $a \in R$, there exists an $S$-submodule $X_a$ of $M$ such that $l_M(r_R(a)) = M_a \oplus X_a$, where $S = \text{End}(M)$. We call $R$ a right AP-injective ring if $R_R$ is an AP-injective module. A ring $R$ is called semiregular$^5$, if for any $a \in R$, there exists an idempotent $g \in Ra$ such that $a(1 - g) \in J(R)$. A ring $R$ is called semi-$\pi$-regular$^6$, if for any $a \in R$, there exist a positive integer $n$ and $e^2 = e \in Ra$ such that $a^n(1 - e) \in J(R)$. We call a ring $R$ right generalized semiregular$^6$, if for any $a \in R$, there exist two left ideals $P, L$ of $R$ such that $lr(a) = P \oplus L$, where $P \subseteq Ra$ and $Ra \cap L$ is small in $R$. A ring $R$ is called GP-injective$^7$, if for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and $lr(a^n) = Ra^n$. From [8, Example 1], we know that GP-injective rings need not be AP-injective, also AP-injective rings need not be GP-injective (See [4, Example 1.5]). Following [6], AP-injective rings and semiregular rings are generalized semiregular, but the converse is not true.

In this paper, we call a ring $R$ right generalized semi-$\pi$-regular, if for any $a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and $lr(a^n) = P \oplus L$, where $P, L$ are left ideals of $R$, $P \subseteq Ra^n$ and $Ra^n \cap L$ is small in $R$. The notion of semi-$\pi$-regular rings is left-right symmetric, but we do

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not know whether this is true for generalized semi-π-regular rings. In this paper, we mainly prove that generalized semi-π-regular rings are non-trivial generalizations of both semi-π-regular rings and GP-injective rings. Also some properties of generalized semi-π-regular rings are studied and some results about GP-injective rings and generalized semiregular rings are extended.

2. Generalized semi-φ-regular rings

Let $R$ be a ring and $M$ a left $R$-module. A submodule $K$ of $M$ is said to be small in $M$, if $K + N \neq M$ for every submodule $N \neq M$.

**Definition 2.1** An element $0 \neq a$ of a ring $R$ is called right generalized semi-π-regular, if there exists a positive integer $n$ such that $a^n \neq 0$ and $lr(a^n) = P \oplus L$, where $P, L$ are left ideals of $R$, $P \subseteq Ra^n$ and $Ra^n \cap L$ is small in $R$. A ring $R$ is called right generalized semi-π-regular if each element is right generalized semi-π-regular. Similarly, we may define left generalized semi-π-regular elements and left generalized semi-π-regular rings.

**Remark 1** By definition, AP-injective rings, GP-injective rings and generalized semiregular rings are generalized semi-π-regular. From [9], we know that a ring $R$ is called right $P$-injective if, for any $a \in R$, $lr(a) = Ra$. Thus every right $P$-injective ring is right generalized semiregular, so it is right generalized semi-π-regular.

**Proposition 2.2** If $R$ is a semi-π-regular ring, then $R$ is right generalized semi-π-regular.

**Proof** Let $0 \neq a \in R$. Since $R$ is semi-π-regular, there exists $e^2 = e \in Ra^n$ for some positive integer $n$ such that $a^n \neq 0$ and $a^n(1 - e) \in J(R)$. Thus $R = Re \oplus R(1 - e)$, where $Re \subseteq Ra^n$ and $Ra^n(1 - e) \subseteq J(R)$ is small in $R$. Note that $Ra^n \not\subseteq lr(a^n)$, so by the modular law we have $lr(a^n) = lr(a^n) \cap R = lr(a^n) \cap (Re \oplus R(1 - e)) = Re \oplus (lr(a^n) \cap R(1 - e))$ and $Ra^n \cap (lr(a^n) \cap R(1 - e)) = Ra^n \cap R(1 - e) \subseteq Ra^n(1 - e) \subseteq J(R)$ is small in $R$. Hence $R$ is right generalized semi-π-regular.

The following two examples show that right generalized semi-π-regular rings are non-trivial generalizations of both GP-injective rings and semi-π-regular rings.

**Example 2.3** Let $R M_R$ be a bimodule over a ring $R$. The trivial extension of $R$ by $M$ is $T(R, M) = R \oplus M$ with pointwise addition and multiplication: $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$. It is shown in [4] that the trivial extension $R = T(Z_4, Z_4 \oplus Z_4)$ is an AP-injective ring, but $R$ is not GP-injective. By Remark 1, $R$ is a generalized semi-π-regular ring.

**Lemma 2.4** If $R$ is a semi-π-regular ring, then $R$ is an exchange ring.

**Example 2.5** Let $R = T(Z, Q/Z)$. By [10], $R$ is a commutative $P$-injective ring, so $R$ is a commutative generalized semi-π-regular ring. But $R/J(R) \cong Z$ is not an exchange ring. So $R$ is not an exchange ring. Thus $R$ is not semi-π-regular ring.

A module $M$ is called AGP-injective [4], if for any $0 \neq a \in R$, there exist a positive integer
n and a S-submodule $X_{a}$ of $M$ such that $a^{n} \neq 0$ and $l_{M}(r_{R}(a^{n})) = Ma^{n} \oplus X_{a}$, where $S = \text{End}(M)$. In each case the module $X_{a}$ may not be unique, but we take one such $X_{a}$ for each $a$ and form the S-module $b(M) = \sum_{a} X_{a}$. We call $b(M)$ an index bound of $M$ and the set of those $X_{a}$ an index set of $M$. We call $R$ a right AGP-injective ring if $R_{R}$ is an AGP-injective module.

By definition, AGP-injective rings are generalized semi-$\pi$-regular, but the converse is not true.

**Lemma 2.6** If $R$ is a right AGP-injective ring, then $J(R) = Z(R_{R})$.

**Example 2.7** Let $R = \left( \begin{array}{cc} Z_{2} & Z_{2} \\ 0 & Z_{2} \end{array} \right)$. Then $J(R) = \left( \begin{array}{cc} 0 & Z_{2} \\ 0 & 0 \end{array} \right)$ and $Z(R_{R}) = Z(R_{R}) = 0$. Thus $R$ is not AGP-injective. But by [6], $R$ is a semiregular ring, so $R$ is a generalized semi-$\pi$-regular ring.

Although generalized semi-$\pi$-regular rings need not be GP-injective or semi-$\pi$-regular, but in the following we will show that right generalized semi-$\pi$-regular rings are GP-injective or semi-$\pi$-regular under some sufficient conditions.

**Proposition 2.8** Let $R$ be a semiprimitive ring. Then $R$ is right generalized semi-$\pi$-regular if and only if $R$ is right AGP-injective.

**Proof** One direction is obvious. Conversely, let $0 \neq a \in R$. Since $R$ is right generalized semi-$\pi$-regular, there exists a positive integer $n$ such that $a^{n} \neq 0$ and $lr(a^{n}) = P \oplus L$, where $P \subseteq Ra^{n}$ and $Ra^{n} \cap L$ is small in $R$. By assumption, $Ra^{n} \cap L \subseteq J(R) = 0$. Clearly, $lr(a^{n}) = Ra^{n} + L$. Thus $lr(a^{n}) = Ra^{n} \oplus L$, which shows that $R$ is right AGP-injective.

**Lemma 2.9** A module $M_{R}$ is GP-injective if and only if $M_{R}$ is AGP-injective with an index bound $b(M) = 0$.

**Corollary 2.10** Let $R$ be a semiprimitive ring with index bound $b(R_{R}) = (0)$. Then $R$ is right generalized semi-$\pi$-regular if and only if $R$ is right GP-injective.

**Proposition 2.11** If $R$ is a right generalized semi-$\pi$-regular ring and for every $0 \neq a^{n} \in R$ there exists $e^{2} = e \in R$ such that $r(a^{n}) = r(e)$, then $R$ is semi-$\pi$-regular.

**Proof** Let $0 \neq a \in R$. Since $R$ is right generalized semi-$\pi$-regular, there exists a positive integer $n$ such that $a^{n} \neq 0$ and $lr(a^{n}) = P \oplus L$, where $P \subseteq Ra^{n}$ and $Ra^{n} \cap L$ is small in $R$. Since $r(a^{n}) = r(e)$, we have $P \oplus L = lr(a^{n}) = lr(e) = Re$. So $a^{n} = a^{n}e$. Let $e = g + t$, where $g = ra^{n} \in P \subseteq Ra^{n}, t \in L$. Then $a^{n} = a^{n}e = a^{n}ra^{n} + a^{n}t$ and $ra^{n} = ra^{n}e = ra^{n}(ra^{n} + t) = ra^{n}ra^{n} + ra^{nt}$. So $ra^{n} - ra^{n}ra^{n} = ra^{nt} \in P \cap L = 0$, which gives $g^{2} = g$ and $a^{n} - a^{n}ra^{n} = a^{nt} \in Ra^{n} \cap L \subseteq J(R)$. This shows that for any $0 \neq a \in R$, there exists $g^{2} = g \in Ra^{n}$ such that $a^{n}(1 - g) \in J(R)$, $a$ is semi-$\pi$-regular. Hence $R$ is semi-$\pi$-regular.

**Lemma 2.12** Let $a \in R$ such that $a^{n}R \cong eR$, where $e^{2} = e \in R$. Then there exists an idempotent $f^{2} = f \in R$ such that $a^{n}f = a^{n}$ and $r(a^{n}) = r(f)$.

**Proof** Let $\sigma : a^{n}R \rightarrow eR$ be the isomorphism. Let $\sigma(a^{n}) = ed, d \in R$ and $\sigma^{-1}(e) = a^{n}c, c \in R$. [4]
Then $edc = \sigma(a^n e) = e$. Take $f = ced$. Then $f^2 = f$ and $a^n f = \sigma^{-1}(ed) = a^n$. Clearly, $r(f) \subseteq r(a^n)$. If $r \in r(a^n)$, then $a^n r = 0$. So $fr = c\sigma(a^n r) = c\sigma(0) = 0$, $r \in r(f)$. This shows that $r(a^n) = r(f)$.

**Proposition 2.13** Let $a$ be a right generalized semi-$\pi$-regular element. If $a^n R \cong e R$, where $e^2 = e \in R$, then $a$ is semi-$\pi$-regular element.

A ring $R$ is called right generalized $P.P.$-ring if, for any $x \in R$ there exists $e^2 = e \in R$ such that $r(a^n) = e R$.

**Corollary 2.14** Let $R$ be a right generalized $P.P.$-ring. If $R$ is a right generalized semi-$\pi$-regular ring, then $R$ is semi-$\pi$-regular.

**Proof** By assumption, for any $a \in R, r(a^n) = e R$, where $e^2 = e \in R$. So $l r(a^n) = l(e R) = l(e) = R(1 - e)$. Let $f = 1 - e$. Then $f^2 = f$ and $l r(a^n) = R f$. Thus $r(a^n) = r l r(a^n) = r(R f) = r(f)$. So $R$ is semi-$\pi$-regular by Proposition 2.11.

A module $M$ is said to satisfy $C_2$ if for any two submodules $X$ and $Y$ of $M$ with $X \cong Y \mid M$, we have $X \mid M$. From [6], we know that if $R$ is a right generalized semiregular ring with $J(R) \subseteq Z(R_R)$, then $R_R$ satisfies $C_2$. But for right generalized semi-$\pi$-regular ring, we only have the following proposition.

**Proposition 2.15** Let $R$ be a right generalized semi-$\pi$-regular ring with $J(R) \subseteq Z(R_R)$. If $e^2 = e \in R$ such that $a^n R \cong e R$, then there exists $g^2 = g \in R$ such that $a^n R = g R$.

**Proof** Let $0 \neq a \in R$ such that $a^n R \cong e R$, where $e^2 = e \in R$. By Lemma 2.12, there exists $f^2 = f \in R$ such that $a^n = a^n f$ and $r(a^n) = r(f)$. By Proposition 2.11, $R$ is semi-$\pi$-regular. So there exists $g^2 = g \in a^n R$ such that $(1 - g)a^n \in J(R)$. Thus $a^n R = g R \bigoplus S$, where $S = (1 - g)a^n R \subseteq J(R)$. By assumption, $S \subseteq Z(R_R)$ is a singular right $R$-module. Let $\varphi$ be the epimorphism of $f R$ to $a^n f R$ given by $\varphi(f r) = a^n f r$ for any $r \in R$. If $a^n f r = 0$, then $fr \in r(a^n) \cap f R = r(f) \cap f R = 0$. So $\varphi$ is isomorphism. This shows that $a^n R = a^n f R \cong f R$ is a projective right $R$-module. Thus $S$ is a projective and singular right $R$-module, and so $S = 0$ by [6, Lemma 2.2]. Hence $a^n R = g R$.

By Lemma 2.6, we know that if $R$ is a right $AGP$-injective ring, then $J(R) = Z(R_R)$.

**Proposition 2.16** If $R$ is a right generalized semi-$\pi$-regular ring, then $Z(R_R) \subseteq J(R)$.

**Proof** Let $0 \neq a \in Z(R_R)$. Then for any $b \in R, ba \in Z(R_R)$. Let $u = 1 - ba$. Then $u \neq 0$. Since $R$ is right generalized semi-$\pi$-regular, there exists a positive integer $n$ such that $u^n \neq 0$ and $lr(u^n) = P \bigoplus L$, where $P \subseteq Ru^n$ and $Ru^n \bigcap L$ is small in $R$. Since $r(ba) \bigcap r(u^n) = 0$, we have $r(u^n) = 0$ and $R = lr(u^n) = P \bigoplus L$. So there exists $e^2 = e \in R$ such that $P = Re$. We claim that $e = 1$. If not, then $(1 - e) R \neq 0$. Since $ba \in Z(R_R), u^n = (1 - ba)^n$, there exists $v \in Z(R_R)$ such that $u^n = 1 - v$. Thus $(1 - e) R \bigcap r(v) = 0$. Let $0 \neq (1 - e) r \in (1 - e) R \bigcap r(v)$. Then $v(1 - e)r = 0$. So $(1 - e)r = u^n(1 - e)r$. Since $u^n \in lr(u^n) = R = Re \bigoplus L$, we take $u^n = se + t$, where $s \in R, t \in L$. Then $(1 - t)(1 - e)r = 0$. Note that $t = u^n - se \in Ru^n \bigcap L \subseteq J(R)$, so
1 − t is a unit, which implies (1 − e)r = 0, a contradiction. So e = 1 and P = Rn. Thus a ∈ J(R).

**Remark 2** Example 2.7 shows that there exists a generalized semi-π-regular ring with J(R) ≠ Z(R)R = Z(RR).

3. Corner subrings of generalized semi-π-regular rings

An idempotent element e ∈ R is left (resp. right) semicentral in R[13] if Re = eRe (resp. eR = eRe). In general we have

**Theorem 3.1** Let R be a right generalized semi-π-regular ring. If e² = e ∈ R is right semicentral, then eRe is right generalized semi-π-regular.

**Proof** Let 0 ≠ a ∈ eRe. By assumption, there exists a positive integer n such that aⁿ ≠ 0 and lr(aⁿ) = P ⊕ L, where P ⊆ Raⁿ and Raⁿ ∩ L is small in R. We claim that l_eRe eRe(aⁿ) = eP ⊕ eL.

In fact, eP ∩ eL ⊆ P ∩ L = 0. Take any y ∈ eP ⊆ ePe, where y = ey₁, y₁ ∈ P ⊆ lr(aⁿ). Then for any x ∈ l_eRe(Raⁿ), y₁x = 0. Thus ey₁x = 0. Hence y ∈ l_eRe eRe(aⁿ), eP ⊆ l_eRe eRe(Raⁿ). Similarly, eL ⊆ l_eRe eRe(aⁿ). Thus eP ⊕ eL ⊆ l_eRe eRe(aⁿ). On the other hand, take any y ∈ l_eRe eRe(aⁿ). Then for any y ∈ l_eRe eRe(aⁿ), eP ⊆ eL is a right generalized semi-π-regular ring. So xey = xey = 0, x ∈ l(y). This shows that l_eRe eRe(aⁿ) ⊆ lr(aⁿ). Let x = s + t, s ∈ P, t ∈ L. Then x = ex = e(s + t) ∈ eP + eL.

Thus l_eRe eRe(aⁿ) = eP ⊕ eL. It remains to prove that eReaⁿ ∩ eL is small in eRe since eP ⊆ eRaⁿ = eReaⁿ. Since e is right semicentral, we have eReaⁿ ∩ eL ⊆ e(eReaⁿ ∩ eL)e. But eReaⁿ ∩ eL ⊆ Raⁿ ∩ l(R), so eReaⁿ ∩ eL ⊆ eJ(R)e = J(eRe). Since J(eRe) is small in eRe, eReaⁿ ∩ eL is small in eRe. Thus eRe is right generalized semi-π-regular.

**Theorem 3.2** Let e² = e ∈ R such that ReR = R. If R is a right generalized semi-π-regular ring, then eRe is right generalized semi-π-regular.

**Proof** Let 0 ≠ a ∈ eRe. By assumption, there exists a positive integer n such that aⁿ ≠ 0 and lr(aⁿ) = P ⊕ L, where P ⊆ Raⁿ and Raⁿ ∩ L is small in R. We claim that l_eRe eRe(aⁿ) = eP ⊕ eLe.

Since 1 − e ∈ r(e) ⊆ r(a) ⊆ r(aⁿ), we have t(1 − e) = 0 for any t ∈ L ⊆ lr(aⁿ). So L = Le. Similarly, P = Pe. Thus ePe ∩ eLe = eP ∩ eL ⊆ P ∩ L = 0. Clearly, ePe ⊆ l_eRe eRe(aⁿ), eLe ⊆ l_eRe eRe(aⁿ). Thus ePe ⊕ eLe ⊆ l_eRe eRe(aⁿ). On the other hand, take any y ∈ l_eRe eRe(aⁿ) and write 1 = ∑ᵢ=₁ⁿ aᵢebᵢ since R = eRe, where aᵢ, bᵢ ∈ R. Then for any y ∈ l_eRe eRe(aⁿ), we have aᵢeyᵢe = aᵢyᵢe = 0, eyᵢe ∈ l_eRe(aⁿ). So xeyᵢe = 0. Thus xy = xey = xey ∑ᵢ=₁ⁿ aᵢebᵢ = ∑ᵢ=₁ⁿ xeyᵢe bᵢ = 0, x ∈ l(y). So l_eRe eRe(aⁿ) ⊆ lr(aⁿ). Let x = s + t, s ∈ P, t ∈ L. Then x = exe = ese + ete ∈ ePe + eLe. Hence l_eRe eRe(aⁿ) = ePe ⊕ eLe. It remains to prove that eReaⁿ ∩ eLe is small in eRe since ePe ⊆ eReaⁿ. Since L = Le, we have eReaⁿ ∩ eLe ⊆ Raⁿ ∩ L ⊆ J(R). So eReaⁿ ∩ eLe ⊆ eJ(R)e = J(eRe). Thus eReaⁿ ∩ eLe is
small in \( eRe \) and \( eRe \) is right generalized semi-\( \pi \)-regular.

**Proposition 3.3** Let \( e \) and \( f \) be orthogonal central idempotents of \( R \). If \( eR \) and \( fR \) are right generalized semi-\( \pi \)-regular, then \( gR = eR \bigoplus fR \) is right generalized semi-\( \pi \)-regular.

**Proof** Let \( 0 \neq a \in gR \). Then \( ea \in eR, fa \in fR \). By assumption, there exists a positive integer \( n \) such that \( a^n \neq 0 \). Take \( x \in l_{eR}f_{gR}(a^n) \). Then for any \( y \in r_{eR}[(ea)^n] \), \( e^na^n y = 0 \). Hence \( a^n y = a^n e^n y = e^n a^n y = 0 \), this implies \( a^n gy = a^n yg = 0 \), \( gy \in r_{gR}(a^n) \). Thus \( xy = xgy = 0 \) and \( exy = xye = 0 \). So \( ex \in l_{eR}r_{eR}[(ea)^n] \). By assumption, \( ex \in l_{eR}r_{eR}[(ea)^n] = P_e \bigoplus L_e \), where \( P_e \subseteq eRea^n = eRa^n, eRa^n \bigcap L_e \subseteq J(eRe) \). Similarly, \( fx \in l_{fR}r_{fR}[(fa)^n] = P_f \bigoplus L_f \), where \( P_f \subseteq fRa^n, fRa^n \bigcap L_f \subseteq J(fRf) \). Then \( x = gx = ex + fx \in P_e \bigoplus P_f \bigoplus L_e \bigoplus L_f \). For any \( x \in L_e \) and any \( y \in r_{gR}(a^n) \), \( a^n y = 0 \), so \( a^n ey = a^n ye = 0 \) and \( xey = 0 \) since \( L_e \subseteq l_{eR}r_{eR}[(ea)^n] \). Note that \( L_e \subseteq eR \subseteq gR \) and \( x = ex, so xy = 0 \) and \( L_e \subseteq l_{gR}r_{gR}(a^n) \). Similarly, \( L_f \subseteq l_{gR}r_{gR}(a^n) \). On the other hand, \( P_e \bigoplus P_f \subseteq eRa^n \bigoplus fRa^n = gRa^n \subseteq l_{gR}r_{gR}(a^n) \). Thus \( l_{gR}r_{gR}(a^n) = P_e \bigoplus P_f \bigoplus L_e \bigoplus L_f \). Since \( gR \) is a ring with identity, \( J(gR) \) is small in \( gR \). We have \( gRa^n \bigcap (L_e \bigoplus L_f) \subseteq J(eR) \bigoplus J(fR) = J(gR) \) is small in \( gR \). Hence \( gR \) is right generalized semi-\( \pi \)-regular.

**Corollary 3.4** Let \( 0 \neq e^2 = e \in R \) be a central idempotent. Then \( eRe \) and \( (1-e)R(1-e) \) are right generalized semi-\( \pi \)-regular if and only if so is \( R \).

**Theorem 3.5** Let \( 1 = e_1 + e_2 + \cdots + e_n \in R \), where \( e_1, e_2, \ldots, e_n \) are orthogonal central idempotents. Then \( R \) is right generalized semi-\( \pi \)-regular if and only if each \( e_i R \) is right generalized semi-\( \pi \)-regular.

**References**


