Journal of Mathematical Research & Exposition
Sept., 2009, Vol. 29, No. 5, pp. 787–798
DOI:10.3770/j.issn:1000-341X.2009.05.003
Http://jmre.dlut.edu.cn

Weighted Polar Decomposition

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Abstract In this paper, a new matrix decomposition called the weighted polar decomposition is considered. Two uniqueness theorems of weighted polar decomposition are presented, and the best approximation property of weighted unitary polar factor and perturbation bounds for weighted polar decomposition are also studied.

Keywords generalized polar decomposition; weighted polar decomposition; weighted unitary polar factor; weighted norm; perturbation bound.

Document code A
MR(2000) Subject Classification 15A09; 15A18; 15A45
Chinese Library Classification O151.2

1. Introduction

Let $C^{m\times n}$, $C_r^{m\times n}$, $C_{\geq}^{m}$, and $C_{>}^{m}$ denote the set of $m\times n$ complex matrices, subset of $C^{m\times n}$ consisting of matrices with rank r, set of Hermitian positive semidefinite matrices of order m, and subset of $C_{\geq}^{m}$ consisting of positive definite matrices, respectively. Given $A\in C^{m\times n}$, the symbols A^*, $r(A)$, $\lambda(A)$, $\lambda_1(A)$, $tr(A)$, A_{MN}^{+}, $R(A)$, $\|A\|_F$, and $\|A\|_2$ stand for the conjugate transpose, rank, nonzero eigenvalues set, biggest eigenvalue, trace, weighted Moore-Penrose inverse, range, Frobenius norm, and spectral norm of A, respectively. In addition, without specification, we always assume that $m\geq n\geq r$ and the given weight matrices $M\in C_{>}^{m}$, $N\in C_{>}^{n}$.

Given the weight matrix M, the weighted inner product in C^m is defined as

$$(x,y)_M = y^*Mx, \quad x,y\in C^m,$$

and the weighted vector norm is defined as

$$\|x\|_M = (x^*Mx)^{1/2} = \|M^{1/2}x\|_2.$$

Moreover, from [1,2], the matrix $X\in C^{n\times m}$ satisfying

$$(Ax,y)_M = (x,Xy)_N, \text{ for all } x\in C^n, y\in C^m$$

is called the weighted conjugate transpose (or adjoint) of the matrix A, and denoted by $X=A^{\#}$. Thus, it is easy to get that

$$A^{\#} = N^{-1}A^*M, \quad A\in C^{m\times n}. \tag{1}$$

The next is the definition of the weighted partial isometric matrix[3,4].

Received date: 2007-10-08; **Accepted date**: 2008-10-06

Definition 1.1 *If $E \in C^{m\times n}$ satisfies*

$$\|Ex\|_M = \|x\|_N\,, \text{ for all } x \in R(E^{\#}),$$

then E is called an (M,N) weighted partial isometric (MN-WPI) matrix. Similarly, if $E \in C^{n\times m}$ satisfies

$$\|Ex\|_N = \|x\|_M\,, \text{ for all } x \in R(E^{\#}),$$

then E is called an (N,M) weighted partial isometric (NM-WPI) matrix. In particular, if $M = I_m$ and $N = I_n$, hereafter I_r denotes the identity matrix of order r, then E is called a partial isometric (or subunitary) matrix[2,5].

Some properties of weighted partial isometric matrices are given in the following lemma[3,4].

Lemma 1.1 *Let $E \in C^{m\times n}$. Then the following statements are equivalent.*

(a) E is an MN-WPI matrix;
(b) $E^{\#}$ is an NM-WPI matrix;
(c) $E^{\#} = E^{+}_{MN}$;
(d) $E^{\#}E$ is an orthogonal projector, i.e., $E^{\#}E = P_{R(E^{\#})}$;
(e) $EE^{\#}$ is an orthogonal projector, i.e., $EE^{\#} = P_{R(E)}$;
(f) $EE^{\#}E = E$;
(g) $E^{\#}EE^{\#} = E^{\#}$.

Let $A \in C_r^{m\times n}$. Then A can be written as

$$A = GE = EH, \tag{2}$$

where $E \in C^{m\times n}$ is an MN-WPI matrix and $MG \in C^{m}_{\geq}$, $NH \in C^{n}_{\geq}$. In this case, G and H are called generalized positive semidefinite matrices. Thus, from (1), we can conclude that G and H are self-adjoint matrices, i.e.,

$$G = G^{\#}, \quad H = H^{\#}. \tag{3}$$

The decomposition (2) is called the (M,N) weighted polar decomposition (MN-WPD)[3] of A, which is a generalization of the (generalized) polar decomposition. The matrices E and G, H are called the (M,N) weighted unitary polar factor and generalized positive semidefinite polar factors of this decomposition, respectively. In general, the MN-WPD is not unique. A unique weighted polar decomposition theorem was proved in [3]. In this paper, we will present two other conditions from which we can also make this decomposition be unique.

If $M = I_m$ and $N = I_n$, then the MN-WPD reduces to the generalized polar decomposition[5,6], and E and G, H reduce to the subunitary polar factor and generalized polar factors, respectively. Further, if $r(A) = n$, then the decomposition is just the polar decomposition and E and H are just the unitary polar factor and positive polar factor.

For the polar decomposition, the best approximation properties and the perturbation bounds of polar factors were studied by Higham[7]. Sun and Chen[6] continued the work of Higham[7] and presented some corresponding results of the generalized polar decomposition. Many other scholars such as Barrlund, Mathias, Li and so on also studied the perturbation bounds for polar decomposition or generalized polar decomposition[8−12], where two perturbation bounds listed in the following for generalized polar decomposition in Frobenius norm are claimed to be optimal in some general sense.

Let $A, \widetilde{A} \in C_r^{m\times n}$ with the generalized polar decompositions:

$$A = EH, \widetilde{A} = \widetilde{E}\widetilde{H},$$

where $\widetilde{A}$ is the perturbed matrix, $E, \widetilde{E}$ are the subunitary polar factors, and $H, \widetilde{H}$ are the generalized polar factors. Then

$$\|\widetilde{H} - H\|_F \leq \sqrt{2}\|\widetilde{A} - A\|_F, \tag{4}$$

$$\|\widetilde{E} - E\|_F \leq \frac{2}{\sigma_r + \widetilde{\sigma}_r}\|\widetilde{A} - A\|_F, \tag{5}$$

where σ_r and $\widetilde{\sigma}_r$ are the smallest singular values of A and $\widetilde{A}$, respectively.

The bound (4) was proved by Sun and Chen[6], while the bound (5) was obtained by Li and Sun[12].

In this paper, we study some properties and perturbation bounds of the MN-WPD. The rest of the paper is organized as follows. Section 2 provides some preliminaries. Section 3 presents two uniqueness theorems of the weighted polar decomposition. Based on the uniqueness theorems, the best approximation property of the weighted unitary polar factor is discussed in Section 4 and the perturbation bounds for weighted polar decomposition are given in Section 5.

2. Preliminaries

In the following, we introduce the definition of the weighted Frobenius norm and the (M, N) singular value decomposition (MN-SVD).

Definition 2.1 *Let $A \in C^{m\times n}$. Then, the norm $\|A\|_{F(MN)} = \|M^{1/2}AN^{-1/2}\|_F$ is called the weighted Frobenius norm of A.*

From Definition 2.1 and the properties of Frobenius norm, the weighted Frobenius norm of A can be expressed as

$$\|A\|_{F(MN)} = \Big(tr((M^{1/2}AN^{-1/2})^*(M^{1/2}AN^{-1/2}))\Big)^{1/2} = (tr(A^{\#}A))^{1/2}. \tag{6}$$

From Van Loan[13], we know the (M, N) singular values of $A \in C_r^{m\times n}$ are the elements of the set $\sigma_{MN}(A)$ defined by

$$\sigma_{MN}(A) = \Big\{\sigma : \sigma \geq 0, \sigma \text{ is a stationary value of } \frac{\|Ax\|_M}{\|x\|_N}\Big\}.$$

By using Lagrange multipliers, for every nonzero element of the set $\sigma_{MN}(A)$, we can get

$$\sigma_i = \lambda_i^{1/2}(N^{-1}A^*MA) = \lambda_i^{1/2}(A^{\#}A), \quad i = 1, \ldots, r.$$

Next is the lemma on MN-SVD which can be found in [1,2,13].

Lemma 2.1 *Let $A \in C_r^{m\times n}$. Then there exist $U \in C^{m\times m}$ and $V \in C^{n\times n}$ satisfying $U^*MU = I_m$*

and $V^*N^{-1}V = I_n$ such that

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{7}$$

where $\Sigma\mathrm{diag}(\sigma_1, \dots, \sigma_r)$, $\sigma_i = \sqrt{\lambda_i}$ *and* $\lambda_1 \geq \cdots \geq \lambda_r > 0$ *are the nonzero eigenvalues of* $N^{-1}A^*MA = A^{\#}A$. *Thus,* $\sigma_1 \geq \cdots \geq \sigma_r > 0$ *are the nonzero* (M, N) *singular values of* A. *In this case, the weighted Moore-Penrose inverse of* A *can be written as*

$$A^+_{MN} = N^{-1}V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*M.$$

Further, let $U = (U_1, U_2)$ *and* $V = (V_1, V_2)$, *where* $U_1 \in C^{m\times r}$ *and* $V_1 \in C^{n\times r}$. *Then*

$$U_1^*MU_1 = I_r, \quad V_1^*N^{-1}V_1 = I_r, \quad A = U_1\Sigma V_1^*, \quad A^+_{MN} = N^{-1}V_1\Sigma^{-1}U_1^*M. \tag{8}$$

3. Uniqueness theorems of weighted polar decomposition

Now we present the first uniqueness theorem of the weighted polar decomposition.

Theorem 3.1 *Let* $A \in C_r^{m\times n}$ *with the MN-WPD in (2). Then the matrices* E, G *and* H *are uniquely determined by*

$$R(E^{\#}) = R(H), \tag{9a}$$

$$R(E) = R(G), \tag{9b}$$

in which case

$$H = (A^{\#}A)^{1/2} = N^{-1}V_1\Sigma V_1^*, \quad G = (AA^{\#})^{1/2} = U_1\Sigma U_1^*M, \quad E = U_1V_1^*, \tag{10}$$

where V_1, U_1, *and* Σ *are as in Lemma 2.1.*

Proof Let E and H satisfy (9a). Then, from (2), (3), and Lemma 1.1, we have

$$A^{\#}A = (EH)^{\#}\, EH = HE^{\#}EH = HP_{R(E^{\#})}H = HP_{R(H)}H = H^2, \tag{11}$$

which proves the uniqueness of H because of the uniqueness of the square root of $A^{\#}A$. The uniqueness of E follows from

$$E = EE^{\#}E = EP_{R(E^{\#})} = EP_{R(H)} = EHH^+_{NN} = AH^+_{NN}.$$

Similarly, (9b) implies $G^2 = AA^{\#}$ and the uniqueness of G. In this case,

$$E = GG^+_{MM}E = M^{-1}\left(G^+_{MM}\right)^* MGE = M^{-1}\left(G^+_{MM}\right)^* MA,$$

which also implies the uniqueness of E.

By (8) and (11), we can get

$$H^2 = A^{\#}A = (U_1\Sigma V_1^*)^{\#}\, U_1\Sigma V_1^* = N^{-1}V_1\Sigma U_1^*MU_1\Sigma V_1^* = N^{-1}V_1\Sigma V_1^*N^{-1}V_1\Sigma V_1^*,$$

which implies

$$H = N^{-1}V_1\Sigma V_1^*,$$

and, consequently,

$$H^{+}_{NN} = N^{-1}V_1\Sigma^{-1}V_1^{*}.$$

Similarly,

$$G = U_1\Sigma U_1^{*}M, \quad G^{+}_{MM} = U_1\Sigma^{-1}U_1^{*}M.$$

Therefore,

$$E = AH^{+}_{NN} = U_1\Sigma V_1^{*}N^{-1}V_1\Sigma^{-1}V_1^{*} = U_1V_1^{*},$$

$$E = M^{-1}\left(G^{+}_{MM}\right)^{*}MA = M^{-1}MU_1\Sigma^{-1}U_1^{*}MU_1\Sigma V_1^{*} = U_1V_1^{*}.$$

Then, we complete the proof. □

In order to give an equivalent condition of (9), namely another uniqueness theorem of the weighted polar decomposition, we need a lemma from [6].

Lemma 3.1 *Let $X \in C^{n\times n}$, $X^{*}X \le I_n$ satisfying $X^{*}\Sigma X = \Sigma$, where $\Sigma = \mathrm{diag}(\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k})$, $\sum_{i=1}^{k} n_i = n$, $\lambda_1 > \cdots > \lambda_k > 0$. Then $X^{*}X = I_n$.*

Theorem 3.2 *Let $A \in C_r^{m\times n}$ with the MN-WPD in (2). Then the following two statements are equivalent.*

(a) $R(E^{\#}) = R(H),\ R(E) = R(G);$ (12)

(b) $r(A) = r(E),\ \lambda(H) = \lambda(G) = \sigma_{MN}(A).$ (13)

Proof (a)⇒(b). From Theorem 3.1, if E, G, and H satisfy (12), then (10) holds. Thus from Lemma 2.1, we have

$$r(A) = r(U_1) = r(E),$$

$$\lambda(H) = \lambda(N^{-1}V_1\Sigma V_1^{*}) = \lambda(\Sigma V_1^{*}N^{-1}V_1) = \lambda(\Sigma) = \sigma_{MN}(A),$$

$$\lambda(G) = \lambda(U_1\Sigma U_1^{*}M) = \lambda(\Sigma U_1^{*}MU_1) = \lambda(\Sigma) = \sigma_{MN}(A).$$

(b)⇒(a). If E, G, and H satisfy (13), then there exist $F_1, Y_1 \in C^{m\times r}$ and $G_1, X_1 \in C^{n\times r}$ satisfying $F_1^{*}MF_1 = Y_1^{*}MY_1 = I_r$ and $G_1^{*}N^{-1}G_1 = X_1^{*}N^{-1}X_1 = I_r$ such that

$$E = F_1G_1^{*}, \quad H = N^{-1}X_1\Sigma X_1^{*}, \quad G = Y_1\Sigma Y_1^{*}M. \tag{14}$$

(i) Firstly, we show that

$$r(A) = r(E), \lambda(H) = \sigma_{MN}(A) \Rightarrow R(E^{\#}) = R(H).$$

By (8), we can conclude that

$$A^{\#}A = N^{-1}V_1\Sigma U_1^{*}MU_1\Sigma V_1^{*} = N^{-1}V_1\Sigma\Sigma V_1^{*}. \tag{15}$$

While, by (2), (3), and (14), we have

$$\begin{aligned} A^{\#}A &= (EH)^{\#}EH = HE^{\#}EH = N^{-1}X_1\Sigma X_1^{*}(F_1G_1^{*})^{\#}F_1G_1^{*}N^{-1}X_1\Sigma X_1^{*} \\ &= N^{-1}X_1\Sigma X_1^{*}N^{-1}G_1G_1^{*}N^{-1}X_1\Sigma X_1^{*}. \end{aligned} \tag{16}$$

Thus, it follows from (15) and (16) that

$$\begin{aligned}N^{-1}V_1\Sigma\Sigma V_1^* &= N^{-1}X_1\Sigma X_1^*N^{-1}G_1G_1^*N^{-1}X_1\Sigma X_1^*\\ &\Leftrightarrow V_1\Sigma^2V_1^* = X_1\Sigma X_1^*N^{-1}G_1G_1^*N^{-1}X_1\Sigma X_1^*. \qquad (17)\end{aligned}$$

Let $C = G_1^*N^{-1}X_1$, $D = X_1^*N^{-1}V_1$, and $W = C\Sigma D\Sigma^{-1}$. Then, by (17), we have

$$\begin{aligned}W^*W &= \Sigma^{-1}V_1^*N^{-1}X_1\Sigma X_1^*N^{-1}G_1G_1^*N^{-1}X_1\Sigma X_1^*N^{-1}V_1\Sigma^{-1}\\ &= \Sigma^{-1}V_1^*N^{-1}(V_1\Sigma^2V_1^*)N^{-1}V_1\Sigma^{-1} = I_r, \qquad (18)\\ D\Sigma^2D^* &= X_1^*N^{-1}V_1\Sigma^2(X_1^*N^{-1}V_1)^*\\ &= X_1^*N^{-1}(X_1\Sigma X_1^*N^{-1}G_1G_1^*N^{-1}X_1\Sigma X_1^*)N^{-1}V_1 = \Sigma C^*C\Sigma. \qquad (19)\end{aligned}$$

Therefore, by (18), we know that W is a unitary matrix, and by (19), we have

$$I_r = W^*W = \Sigma^{-1}D^*\Sigma C^*C\Sigma D\Sigma^{-1} = \Sigma^{-1}D^*(D\Sigma^2D^*)D\Sigma^{-1},$$

i.e.,

$$\Sigma^2 = D^*D\Sigma^2D^*D. \qquad (20)$$

From Lemma 3.1 and (20), we know D^*D is a unitary matrix, i.e., $(D^*D)^2 = I_r$, which together with $D^*D \geq 0$ implies $D^*D = I_r$, namely D is a unitary matrix. Meanwhile, since

$$(C^{-1}W)\Sigma^2(C^{-1}W)^* = \Sigma D\Sigma^{-1}\Sigma^2\Sigma^{-1}D^*\Sigma = \Sigma DD^*\Sigma = \Sigma^2,$$

from Lemma 3.1, we know that $C^{-1}W$ is a unitary matrix. Thus, C is also a unitary matrix. Therefore,

$$CC^* = G_1^*N^{-1}X_1C^* = I_r.$$

Let $\widetilde{X}_1 = X_1C^*$. Then

$$G_1^*N^{-1}\widetilde{X}_1 = I_r, \quad G_1^*N^{-1}G_1 = I_r, \quad \widetilde{X}_1^*N^{-1}\widetilde{X}_1 = CX_1^*N^{-1}X_1C^* = I_r. \qquad (21)$$

Thus, let $G = (G_1, G_2)$ and $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$ satisfy $G^*N^{-1}G = I_n$ and $\widetilde{X}^*N^{-1}\widetilde{X} = I_n$. Then, by (21), we have

$$G^*N^{-1}\widetilde{X} = \begin{pmatrix} I_r & G_1^*N^{-1}\widetilde{X}_2 \\ G_2^*N^{-1}\widetilde{X}_1 & G_2^*N^{-1}\widetilde{X}_2 \end{pmatrix}. \qquad (22)$$

Observe that $G^*N^{-1}\widetilde{X}$ is a unitary matrix. In fact,

$$(G^*N^{-1}\widetilde{X})^*G^*N^{-1}\widetilde{X} = \widetilde{X}^*N^{-1}GG^*N^{-1}\widetilde{X} = \widetilde{X}^*N^{-1}GG^*N^{-1}GG^{-1}\widetilde{X} = I_n,$$

as a result, $G_2^*N^{-1}\widetilde{X}_1 = G_1^*N^{-1}\widetilde{X}_2 = 0$. Then, by (22), we can get

$$\widetilde{X} = (G^*N^{-1})^{-1}\begin{pmatrix} I_r & 0 \\ 0 & G_2^*N^{-1}\widetilde{X}_2 \end{pmatrix} = G\begin{pmatrix} I_r & 0 \\ 0 & G_2^*N^{-1}\widetilde{X}_2 \end{pmatrix}.$$

Consequently, $\widetilde{X}_1 = G_1 = X_1C^*$, which together with (14) gives

$$R(E^\#) = R(N^{-1}G_1F_1^*M) = R(N^{-1}G_1) = R(N^{-1}X_1C^*) = R(N^{-1}X_1) = R(H).$$

(ii) To prove $r(A)=r(E)$, $\lambda(G)=\sigma_{MN}(A)\Rightarrow R(E)=R(G)$, we only let C, D, and W in (i) be replaced by $C=F_1^*MY_1$, $D=Y_1^*MU_1$, and $W=C\Sigma D\Sigma^{-1}$. Here, we omit the detail. $\square$

4. Best approximation property of weighted unitary polar factor

In this section, we study the best approximation property of the weighted unitary polar factor using MN-SVD. Firstly, we present a lemma.

Lemma 4.1 *Let $A\in C_r^{m\times n}$ with the MN-SVD as Lemma 2.1. Then*

$$\max_{\widetilde{E}\in U_r^{m\times n}} \mathrm{Re}tr(\widetilde{E}^{\#}A)=\sum_{i=1}^{r}\sigma_i=\mathrm{Re}tr(E^{\#}A),$$

where $U_r^{m\times n}$ denotes the set of $m\times n$ MN-WPI matrices with rank r, $E\in C^{m\times n}$ is the (M,N) weighted unitary polar factor of A given in (10), $\sigma_1\geq\cdots\geq\sigma_r>0$ are the (M,N) singular values of A, and Re stands for the real part of a complex number.

Proof For any $\widetilde{E}\in U_r^{m\times n}$, let $W=V_1^*\widetilde{E}^{\#}U_1=(w_{ij})$. Then $\|W\|_2\leq 1$. In fact, from the definition of the spectral norm, Lemma 1.1, Poincare separation theorem[14], and (8), we can get

$$\begin{aligned}
\|W\|_2&=\lambda_1^{1/2}(W^*W)=\lambda_1^{1/2}((V_1^*\widetilde{E}^{\#}U_1)^*V_1^*\widetilde{E}^{\#}U_1)\\
&=\lambda_1^{1/2}((M^{1/2}U_1)^*M^{1/2}\widetilde{E}N^{-1}VV_1^*N^{-1}\widetilde{E}^*M^{1/2}M^{1/2}U_1)\\
&\leq\lambda_1^{1/2}(M^{1/2}\widetilde{E}N^{-1}VV_1^*N^{-1}\widetilde{E}^*M^{1/2})\\
&=\lambda_1^{1/2}((N^{-1/2}V_1)^*N^{-1/2}\widetilde{E}^*M\widetilde{E}N^{-1/2}N^{-1/2}V)\\
&\leq\lambda_1^{1/2}(N^{-1/2}\widetilde{E}^*M\widetilde{E}N^{-1/2})=\lambda_1^{1/2}(\widetilde{E}^{\#}\widetilde{E})=1,
\end{aligned}$$

which gives $|w_{ii}|\leq 1$, $i=1,2,\ldots,r$. Then

$$\mathrm{Re}tr(\widetilde{E}^{\#}A)=\mathrm{Re}tr(\widetilde{E}^{\#}U_1\Sigma V_1^*)=\mathrm{Re}tr(V_1^*\widetilde{E}^{\#}U_1\Sigma)=\mathrm{Re}tr(W\Sigma)\leq\sum_{i=1}^{r}\sigma_i, \tag{23}$$

where the equality holds if and only if $w_{ii}=1$, $i=1,2,\ldots,r$. Meanwhile, it is easy to get that when $\|W\|_2\leq 1$, then $w_{ii}=1, i=1,2,\ldots,r$ if and only if $W=I_r$. Therefore, the equality in (23) holds if and only if $W=I_r$, i.e., $V_1^*\widetilde{E}^{\#}U_1=I_r$.

Since, for any $\widetilde{E}\in U_r^{m\times n}$, there must exist $X_1=\begin{array}{cc}\begin{pmatrix}X_{11}\\X_{21}\end{pmatrix} & \begin{array}{c}r\\m-r\end{array}\\ r & \end{array}$ and $Y_1=\begin{array}{cc}\begin{pmatrix}Y_{11}\\Y_{21}\end{pmatrix} & \begin{array}{c}r\\n-r\end{array}\\ r & \end{array}$ satisfying $X_1^*X_1=Y_1^*Y_1=I_r$ such that $\widetilde{E}=UX_1Y_1^*V^*$. Using the methods in [6], we can prove that $V_1^*\widetilde{E}^{\#}U_1=I_r$ if and only if $X_{11}=Y_{11}$ are unitary matrices. In fact, the sufficiency is trivial. We prove the necessity in the following. From

$$\begin{aligned}
I_r&=V_1^*\widetilde{E}^{\#}U_1=V_1^*(UX_1Y_1^*V^*)^{\#}U_1=V_1^*N^{-1}VY_1X_1^*U^*MU_1\\
&=V_1^*N^{-1}(V_1,V_2)Y_1X_1^*(U_1,U_2)^*MU_1=I_rY_1X_1^*I_r=Y_{11}X_{11}^*,
\end{aligned}$$

we have $Y_{11}=(X_{11}^*)^{-1}$, which together with $X_{11}^*X_{11}\leq I_r$ and

$$Y_{11}^*Y_{11}=X_{11}^{-1}(X_{11}^*)^{-1}=(X_{11}^*X_{11})^{-1}\leq I_r$$

gives $X_{11}^*X_{11} = I_r$, namely, X_{11} is a unitary matrix. Then $Y_{11} = (X_{11}^*)^{-1}$ is also a unitary matrix. Consequently, we complete the proof of the necessity. In this case, we have $X_{21} = 0$, $Y_{21} = 0$, i.e.,

$$X_1 = \begin{pmatrix} X_{11} \\ 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} Y_{11} \\ 0 \end{pmatrix}.$$

Therefore,

$$\widetilde{E} = UX_1Y_1^*V^* = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}V^* = U_1V_1^* = E.$$

That is to say that $W = I_r$ if and only if $\widetilde{E} = E$. Then

$$\max_{\widetilde{E}\in U_r^{m\times n}} \mathrm{Re}tr(\widetilde{E}^{\#}A) = \sum_{i=1}^{r}\sigma_i = \mathrm{Re}tr(E^{\#}A),$$

and $E \in C^{m\times n}$ is the only matrix that makes the equality hold. □

Theorem 4.1 *Let $A \in C_r^{m\times n}$ with the MN-WPD in (2). Then*

$$\min_{\widetilde{E}\in U_r^{m,n}} \|A-\widetilde{E}\|_{F(MN)} = \|A-E\|_{F(MN)} = \Big(\sum_{i=1}^{r}(\sigma_i-1)^2\Big)^{1/2}, \tag{24}$$

where $U_r^{m\times n}$, E, and σ_i $(i = 1, 2, \ldots, r)$ are as in Lemma 4.1.

Proof It follows from (6) and Lemma 4.1 that

$$\begin{aligned}\|A-\widetilde{E}\|_{F(MN)}^2 &= tr\Big((A-\widetilde{E})^{\#}(A-\widetilde{E})\Big) = tr(A^{\#}A) + tr(\widetilde{E}^{\#}\widetilde{E}) - 2\mathrm{Re}tr(\widetilde{E}^{\#}A)\\ &\geq tr(A^{\#}A) + tr(E^{\#}\widetilde{E}) - 2\mathrm{Re}tr(E^{\#}A) = \|A-E\|_{F(MN)}^2,\end{aligned}$$

where the equality holds if and only if $\widetilde{E} = E = U_1V_1^*$, and E is the only matrix that makes the equality hold. Moreover,

$$\|A-E\|_{F(MN)}^2 = tr\left((A-E)^{\#}(A-E)\right) = tr\left(A^{\#}A + E^{\#}E - A^{\#}E - E^{\#}A\right).$$

Meanwhile, from (8) and $E = U_1V_1^*$, we have

$$\begin{aligned}&tr\left(A^{\#}A + E^{\#}E - A^{\#}E - E^{\#}A\right)\\ &= tr\left((U_1\Sigma V_1^*)^{\#}U_1\Sigma V_1^* + (U_1V_1^*)^{\#}U_1V_1^* - (U_1\Sigma V_1^*)^{\#}U_1V_1^* - (U_1V_1^*)^{\#}U_1\Sigma V_1^*\right)\\ &= tr\left(N^{-1}V_1\Sigma\Sigma V_1^* + N^{-1}V_1V_1^* - N^{-1}V_1\Sigma V_1^* - N^{-1}V_1\Sigma V_1^*\right)\\ &= tr\left(\Sigma^2 - 2\Sigma + I_r\right) = \sum_{i=1}^{r}(\sigma_i-1)^2.\end{aligned}$$

Therefore, (24) holds. □

5. Perturbation bounds

We begin this section with two lemmas from [9] and [12], respectively.

Lemma 5.1 *Let $W \in C^{n\times n}$ be a unitary matrix, $X \in C^{n\times n}$ satisfy $\|X\|_2 \leq 1$, and $\Sigma =$*

$\operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, $\widetilde{\Sigma} = \operatorname{diag}(\widetilde{\sigma}_1, \widetilde{\sigma}_2, \dots, \widetilde{\sigma}_n)$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, $\widetilde{\sigma}_1 \geq \widetilde{\sigma}_2 \geq \cdots \geq \widetilde{\sigma}_n \geq 0$. *Then*

$$|tr(\Sigma X^* \widetilde{\Sigma} W)| \leq \frac{1}{2}\mathrm{Re}tr(\Sigma W^* \widetilde{\Sigma} W) + \frac{1}{2}\sum_{i=1}^{n} \sigma_i \widetilde{\sigma}_i.$$

Remark 5.1 From the proof of Lemma 2.3 in [9], we can get that, in fact, the unitary matrix W in Lemma 5.1 can be replaced by any $W \in C^{n\times n}$.

Lemma 5.2 *Let $B \in C^{m\times m}$ and $C \in C^{n\times n}$ be two unitary matrices, and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$, $\widetilde{\Sigma} = \operatorname{diag}(\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_r)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, $\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_r > 0$. Then*

$$\left\| B \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} C \right\|_F \geq \frac{\sigma_r + \widetilde{\sigma}_r}{2} \left\| B \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} C \right\|_F.$$

Next we discuss the perturbation bounds for the weighted unitary polar factor and the generalized positive semidefinite polar factors of the MN-WPD in the weighted Frobenius norm.

Theorem 5.1 *Let $A, \widetilde{A} \in C_r^{m\times n}$, and $A = GE = EH$, $\widetilde{A} = \widetilde{G}\widetilde{E} = \widetilde{E}\widetilde{H}$ be the MN-WPDs of A and $\widetilde{A}$ satisfying the condition (12) or (13). Then*

$$\|\widetilde{H} - H\|_{F(NN)} \leq \sqrt{2}\|\widetilde{A} - A\|_{F(MN)}, \tag{25}$$

$$\|\widetilde{G} - G\|_{F(MM)} \leq \sqrt{2}\|\widetilde{A} - A\|_{F(MN)}, \tag{26}$$

$$\|\widetilde{E} - E\|_{F(MN)} \leq \frac{2}{\sigma_r + \widetilde{\sigma}_r}\|\widetilde{A} - A\|_{F(MN)}, \tag{27}$$

where σ_r and $\widetilde{\sigma}_r$ are the smallest (M, N) singular values of A and $\widetilde{A}$, respectively.

Proof Similarly to Lemma 2.1, let $\widetilde{A}$ have the MN-SVDs:

$$\widetilde{A} = \widetilde{U} \begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \widetilde{V}^*, \tag{28}$$

and let $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2)$, $\widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2)$, where $\widetilde{U}_1 \in C^{m\times r}$, $\widetilde{V}_1 \in C^{n\times r}$. Thus, from Theorem 3.1, we have

$$\widetilde{A} = \widetilde{G}\widetilde{E} = \widetilde{E}\widetilde{H}, \tag{29}$$

where

$$\widetilde{E} = \widetilde{U}_1\widetilde{V}_1^*, \quad \widetilde{G} = \widetilde{U}_1\widetilde{\Sigma}\widetilde{U}_1^* M, \quad \widetilde{H} = N^{-1}\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*. \tag{30}$$

(i) Since the ways to prove the perturbation bounds for H and G are similar, here, we only prove (25). By (6), (3), (8), and (30), we have

$$\begin{aligned} \|\widetilde{H} - H\|_{F(NN)}^2 &= tr((\widetilde{H} - H)^{\#}(\widetilde{H} - H)) = tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(H\widetilde{H}) \\ &= tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(N^{-1}V_1\Sigma V_1^* N^{-1}\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*). \end{aligned} \tag{31}$$

Let $W = \widetilde{V}_1^* N^{-1} V_1 \in C^{r\times r}$. Then (31) can be rewritten as

$$\|\widetilde{H} - H\|_{F(NN)}^2 = tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(\Sigma W^* \widetilde{\Sigma} W). \tag{32}$$

While, from (6), (2), Theorem 3.1, (29), and (30), we have

$$\begin{aligned}
\|\widetilde{A}-A\|_{F(MN)}^2 &= tr((\widetilde{E}\widetilde{H}-EH)^{\#}(\widetilde{E}\widetilde{H}-EH)) \\
&= tr(\widetilde{H}\widetilde{E}^{\#}\widetilde{E}\widetilde{H}) + tr(HE^{\#}EH) - 2\mathrm{Re}tr(HE^{\#}\widetilde{E}\widetilde{H}) \\
&= tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(HE^{\#}\widetilde{E}\widetilde{H}) \\
&= tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(N^{-1}V_1\Sigma V_1^* N^{-1}V_1U_1^*M\widetilde{U}_1\widetilde{V}_1^*N^{-1}\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*) \\
&= tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(N^{-1}V_1\Sigma U_1^*M\widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^*). \qquad (33)
\end{aligned}$$

Let $X=\widetilde{U}_1^*MU_1\in C^{r\times r}$. Then (33) can be rewritten as

$$\|\widetilde{A}-A\|_{F(MN)}^2 = tr(\widetilde{H}^2) + tr(H^2) - 2\mathrm{Re}tr(\Sigma X^*\widetilde{\Sigma}W). \qquad (34)$$

Note that (from Lemma 2.1 and Poincare separation theorem[14]),

$$\|X\|_2^2 = \|\widetilde{U}_1^*MU_1\|_2^2 = \lambda_1(U_1^*M^{1/2}M^{1/2}\widetilde{U}_1\widetilde{U}_1^*M^{1/2}M^{1/2}U_1) \le \lambda_1(M^{1/2}\widetilde{U}_1\widetilde{U}_1^*M^{1/2}) = 1.$$

Then $X\in C^{r\times r}$ satisfies $\|X\|_2\le 1$. Thus, it follows from Lemma 5.1, Remark 5.1, (32), and (34) that

$$\begin{aligned}
\|\widetilde{A}-A\|_{F(MN)}^2 &\ge tr(\widetilde{H}^2) + tr(H^2) - 2|tr(\Sigma X^*\widetilde{\Sigma}W)| \\
&\ge tr(\widetilde{H}^2) + tr(H^2) - \mathrm{Re}tr(\Sigma W^*\widetilde{\Sigma}W) - \sum_{i=1}^r \sigma_i\widetilde{\sigma}_i \\
&= \frac{1}{2}\|\widetilde{H}-H\|_{F(NN)}^2 + \frac{tr(\widetilde{H}^2)+tr(H^2)-2\sum\limits_{i=1}^r\sigma_i\widetilde{\sigma}_i}{2} \\
&= \frac{1}{2}\|\widetilde{H}-H\|_{F(NN)}^2 + \frac{tr(\widetilde{\Sigma}^2)+tr(\Sigma^2)-2\sum\limits_{i=1}^r\sigma_i\widetilde{\sigma}_i}{2} \ge \frac{1}{2}\|\widetilde{H}-H\|_{F(NN)}^2.
\end{aligned}$$

Therefore,

$$\|\widetilde{H}-H\|_{F(NN)} \le \sqrt{2}\|\widetilde{A}-A\|_{F(MN)}.$$

(ii) Now we prove the perturbation bound (27). From Definition 2.1, (7), and (28), we have

$$\begin{aligned}
\|\widetilde{A}-A\|_{F(MN)} &= \|M^{1/2}(\widetilde{A}-A)N^{-1/2}\|_F \\
&= \left\| M^{1/2}\widetilde{U}\begin{pmatrix}\widetilde{\Sigma} & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^*N^{-1/2} - M^{1/2}U\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^*N^{-1/2}\right\|_F \\
&= \left\| U^*M\widetilde{U}\begin{pmatrix}\widetilde{\Sigma} & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}V^*N^{-1}\widetilde{V}\right\|_F. \qquad (35)
\end{aligned}$$

Meanwhile,

$$U^*M\widetilde{U}(U^*M\widetilde{U})^* = U^*M\widetilde{U}\widetilde{U}^*MU = U^*M\widetilde{U}\widetilde{U}^*M\widetilde{U}\widetilde{U}^{-1}U = I_m.$$

Similarly,

$$V^*N^{-1}\widetilde{V}(V^*N^{-1}\widetilde{V})^* = I_n.$$

Both of which imply that $U^*M\widetilde{U} \in C^{m\times m}$ and $V^*N^{-1}\widetilde{V} \in C^{n\times n}$ are unitary matrices. Then, it follows from Lemma 5.2 and (35) that

$$\|\widetilde{A}-A\|_{F(MN)} \geq \frac{\sigma_r+\widetilde{\sigma}_r}{2}\left\|U^*M\widetilde{U}\begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix} - \begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix}V^*N^{-1}\widetilde{V}\right\|_F. \tag{36}$$

Notice that

$$U^*MU = I_m \Leftrightarrow (M^{1/2}U)^*M^{1/2}U = I_m \Leftrightarrow U^*M^{1/2} = (M^{1/2}U)^{-1},$$
$$\widetilde{V}^*N^{-1}\widetilde{V} = I_n \Leftrightarrow (N^{-1/2}\widetilde{V})^*N^{-1/2}\widetilde{V} = I_n \Leftrightarrow \widetilde{V}^*N^{-1/2} = (N^{-1/2}\widetilde{V})^{-1}.$$

Then from (36), we have

$$\|\widetilde{A}-A\|_{F(MN)} \geq \frac{\sigma_r+\widetilde{\sigma}_r}{2}\left\|M^{1/2}\widetilde{U}\begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix}\widetilde{V}^*N^{-1/2} - M^{1/2}U\begin{pmatrix} I_r & 0 \\ 0 & 0\end{pmatrix}V^*N^{-1/2}\right\|_F$$

$$= \frac{\sigma_r+\widetilde{\sigma}_r}{2}\|M^{1/2}\widetilde{U}_1\widetilde{V}_1^*N^{-1/2} - M^{1/2}U_1V_1^*N^{-1/2}\|_F$$

$$= \frac{\sigma_r+\widetilde{\sigma}_r}{2}\|M^{1/2}\widetilde{E}N^{-1/2} - M^{1/2}EN^{-1/2}\|_F = \frac{\sigma_r+\widetilde{\sigma}_r}{2}\|\widetilde{E}-E\|_{F(MN)},$$

i.e.,

$$\|\widetilde{E}-E\|_{F(MN)} \leq \frac{2}{\sigma_r+\widetilde{\sigma}_r}\|\widetilde{A}-A\|_{F(MN)}.$$

Thus, we complete the proof. □

Remark 5.2 From (25), we can get that

$$\lim_{\widetilde{A}\to A}\|\widetilde{H}-H\|_{F(NN)} = 0.$$

That is to say the generalized positive semidefinite polar factor of the weighted polar decomposition is continuous and then is always well behaved under perturbations. Whereas, in general, the weighted unitary polar factor of the weighted polar decomposition is discontinuous. Here, we take an example from [6] to verify it.

Example 5.1 Let A and $\widetilde{A}$ have the MN-SVDs:

$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0\end{pmatrix}V^*, \quad \widetilde{A} = U\begin{pmatrix} \Sigma & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0\end{pmatrix}V^*,$$

where $\varepsilon > 0$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, and $U \in C^{m\times m}$ and $V \in C^{n\times n}$ satisfying $U^*MU = I_m$, $V^*N^{-1}V = I_n$.

Let A and $\widetilde{A}$ have the MN-WPDs as (2) satisfying the condition (12) or (13), and let $U = (U_1, u_{r+1}, \ldots, u_m)$ and $V = (V_1, v_{r+1}, \ldots, v_n)$. Then, the weighted unitary polar factors of A and $\widetilde{A}$ can be obtained from Theorem 3.1, i.e.,

$$E = U_1V_1^*, \quad \widetilde{E} = (U_1, u_{r+1})(V_1, v_{r+1})^* = E + u_{r+1}v_{r+1}^*.$$

Therefore,

$$\|\widetilde{A} - A\|_{F(MN)} = \|M^{1/2}u^{r+1}\varepsilon v_{r+1}^{*}N^{-1/2}\|_F = \varepsilon,$$
$$\|\widetilde{E} - E\|_{F(MN)} = \|M^{1/2}u_{r+1}v_{r+1}^{*}N^{-1/2}\|_F = 1,$$

namely, $\widetilde{E}$ is discontinuous.

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