

Riordan Arrays and Some Identities Containing the Genocchi Numbers

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Abstract In this paper, using generating functions and Riordan arrays, we get some identities relating Genocchi numbers with Stirling numbers and Cauchy numbers.

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1. Introduction

The Genocchi numbers G_n are usually defined by means of the following generating function

$$\frac{2t}{e^t + 1} = \sum_{n \geq 1} G_n \frac{t^n}{n!} \quad (1)$$

and $G_0 := 0$. The relationships of the Genocchi numbers G_n with the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ are known as follows^[1]

$$G_{2n} = 2(1 - 2^{2n})B_{2n} \quad (2)$$

and

$$G_n = nE_{n-1}(0) \quad \text{for } n \geq 1, \quad (3)$$

where the Bernoulli numbers B_n and the Euler polynomial $E_n(x)$ are defined by the generating functions

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!} \quad (4)$$

and

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n \geq 0} E_n(x) \frac{t^n}{n!} \quad (5)$$

respectively. We can get the following recurrence relation:

$$G_n = n! \delta_{n1} - \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} G_k, \quad n \geq 1, \quad (6)$$

because

$$\sum_{k=0}^n \frac{G_k}{k!} \frac{1}{(n-k)!} + \frac{G_n}{n!} = [t^n] \frac{2t}{e^t + 1} (e^t + 1) = [t^n] 2t = 2\delta_{n1}.$$

Thus we have the first values of the Genocchi numbers

n	1	2	4	6	8	10	12	14
G_n	1	-1	1	-3	17	-155	2073	-38227

Dumont^[2] gave the combinatorial interpretations of the Genocchi numbers in 1970s. From then on, Dumont and some collaborators^[3-6] studied the Genocchi numbers. Dumont showed that the Genocchi number $(-1)^n G_{2n}$ is the number of permutations of $[2(n-1)]$ such that each even integer must be followed by a smaller integer (in particular, the sequence cannot end with an even integer) and each odd integer is either followed by a larger integer or is final in the sequence. Recently, new researches of this field have arisen^[7-10].

In the present paper we give some identities related to the Genocchi numbers in terms of Riordan arrays. The proofs are very short. The concept of the Riordan array was first introduced by Shapiro et al.^[11]. It is important in studying combinatorial identities and combinatorial sums. For example, using Riordan arrays, we can find the generating function of many combinatorial sums. Moreover, we can find a closed form or asymptotic value for the sums. In 1994, Sprugnoli^[12] studied Riordan arrays related to binomial coefficients, coloured walks and Stirling numbers. In 1995, Sprugnoli^[13] studied the identities of Abel and Gould by Riordan arrays. In 2003, Zhao and Wang^[14] used the concept of Riordan array on reciprocal functions and gave some identities involving binomial numbers, Stirling numbers and many other special numbers. Recently, Merlini et al.^[15] studied many properties of Cauchy numbers in terms of generating functions and Riordan arrays.

Let $\mathbb{R}[[t]]$ be the ring of formal power series in some indeterminate t . For a sequence $(f_k)_{k \in \mathbb{N}}$, the formal power series $f(t) = \sum_{k=0}^{\infty} f_k \frac{t^k}{k!}$ is called the exponential generating function, and we write $f(t) = \mathcal{E}_t(f_k)_{k \in \mathbb{N}} = \mathcal{E}(f_k)_{k \in \mathbb{N}}$. If $f(t) \in \mathbb{R}[[t]]$, $f_k = [t^k]f(t)$ denotes the coefficient of t^k in the expansion of $f(t)$ in t .

An exponential Riordan array is an infinite lower triangular array $D = (d_{n,k})_{n,k \in \mathbb{N}}$ defined by a couple of formal power series: $D = (d_{n,k}) = (d(t), h(t))_E$, such that

$$d_{n,k} = \left[\frac{t^n}{n!} \right] d(t) \frac{(th(t))^k}{k!}, \quad \forall n \in \mathbb{N}. \quad (7)$$

For example, the pascal triangle $\left(\binom{n}{k} \right)_{n,k \in \mathbb{N}}$ is an exponential Riordan array $(e^t, 1)_E$, because

$$\binom{n}{k} = \left[\frac{t^n}{n!} \right] e^t \frac{t^k}{k!}.$$

The most important property of Riordan arrays is: If $D = (d(t), h(t))_E$ is an exponential Riordan array and $f(t)$ is the exponential generating function of the sequence $(f_k)_{k \in \mathbb{N}}$, then we have

$$\sum_{k=0}^{\infty} d_{n,k} f_k = \left[\frac{t^n}{n!} \right] d(t) f(th(t)) = \left[\frac{t^n}{n!} \right] d(t) [f(y) \mid y = th(t)], \quad (8)$$

where we use the notation $[f(y) \mid y = g(t)]$ as a linearization of the more common one $f(y) \mid_{y=g(t)}$ to denote substitution $f(g(t))$.

This paper is organized as follows. Using Riordan arrays, we give some identities involving the Genocchi numbers and the Stirling numbers in Section 2, and give the closed form or asymptotic value of some double sums related to the Genocchi numbers and the Cauchy numbers in Section 3.

2. Genocchi numbers and Stirling numbers

We will meet two kinds of Stirling numbers. The Stirling numbers of the first kind $(-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ are defined by the following exponential generating function

$$\mathcal{E} \left((-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \right)_n = \frac{(\log(1+t))^k}{k!}, \tag{9}$$

and the unsigned Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ are defined by

$$\mathcal{E} \left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \right)_n = \frac{(\log \frac{1}{1-t})^k}{k!}. \tag{10}$$

The Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$, which are the numbers of distributions of n distinct balls into k indistinguishable boxes (the order of the boxes does not count) such that no box is empty, have the following exponential generating function

$$\mathcal{E} \left(\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \right)_n = \frac{(e^t - 1)^k}{k!}. \tag{11}$$

By definition (7), we have three exponential Riordan arrays:

$$\left((-1)^{n-k} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \right)_E = \left(1, \frac{1}{t} \log(1+t) \right)_E, \tag{12}$$

$$\left(\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \right)_E = \left(1, \frac{1}{t} \log \frac{1}{1-t} \right)_E \tag{13}$$

and

$$\left(\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \right)_E = \left(1, \frac{e^t - 1}{t} \right)_E. \tag{14}$$

Using the exponential Riordan array (14) and formula (8), we can get Genocchi numbers from the Stirling numbers of the second kind $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

Theorem 1^[16, p 54] For $n \in \mathbb{N}$, the following identity holds true

$$\sum_{k=0}^n k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left(-\frac{1}{2} \right)^k = \frac{G_{n+1}}{n+1}. \tag{15}$$

Proof Because the exponential generating function of the sequence $k!(-\frac{1}{2})^k$ is $\frac{1}{1+\frac{t}{2}}$, we have

$$\begin{aligned} \sum_{k=0}^n k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left(-\frac{1}{2} \right)^k &= \left[\begin{smallmatrix} t^n \\ n! \end{smallmatrix} \right] \left[\frac{1}{1+\frac{y}{2}} \mid y = e^t - 1 \right] = \left[\begin{smallmatrix} t^n \\ n! \end{smallmatrix} \right] \frac{2}{1+e^t} \\ &= \frac{1}{n+1} \left[\begin{smallmatrix} t^{n+1} \\ (n+1)! \end{smallmatrix} \right] \frac{2t}{1+e^t} = \frac{G_{n+1}}{n+1}. \quad \square \end{aligned}$$

Corollary 2 For $n \in \mathbb{N}$, there holds

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{G_{k+1}}{k+1} = \frac{n!}{2^n}. \tag{16}$$

Proof Using the inverse relation

$$a_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} b_k \iff b_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} a_k, \tag{17}$$

we get $n!(-\frac{1}{2})^n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{G_{k+1}}{k+1}$. The identity follows immediately. \square

From the exponential Riordan array (13) and formula (8), we have the following identities.

Theorem 3 For $n \geq 1$, there holds

$$\sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} G_m = 2(n-1)! - n! \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{2}\right)^{n-k}. \tag{18}$$

Proof It is well known that

$$[t^n] \frac{\log(1-t)}{t} = -\frac{1}{n+1} \quad \text{and} \quad [t^n] \log \frac{1}{1-t} = \frac{1}{n}.$$

Thus we have

$$\begin{aligned} \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix} G_m &= \left[\frac{t^n}{n!} \left[\frac{2y}{1+e^y} \mid y = \log \frac{1}{1-t} \right] \right] \\ &= n! [t^n] \frac{2(1-t)}{2-t} \log \frac{1}{1-t} = 2n! [t^n] \log \frac{1}{1-t} + n! [t^n] \frac{1}{1-\frac{t}{2}} \log(1-t) \\ &= 2n! [t^n] \log \frac{1}{1-t} + n! [t^{n-1}] \frac{1}{1-\frac{t}{2}} \frac{\log(1-t)}{t} \\ &= 2(n-1)! - n! \sum_{k=1}^n \frac{1}{k} \left(\frac{1}{2}\right)^{n-k}, \end{aligned}$$

where $n \geq 1$. \square

Similarly, using the exponential Riordan array (12) and formula (8), we obtain

Theorem 4 For $n \geq 1$, we have

$$\sum_{m=0}^n (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} G_m = (-1)^{n-1} n! \sum_{l=1}^n \frac{1}{l2^{n-l}}. \tag{19}$$

Theorem 5 For $n \geq 1$ and $m \geq 1$, the following identity holds

$$\frac{1}{m+1} \sum_{k=0}^n \binom{m+k}{k} \begin{bmatrix} n \\ m+k \end{bmatrix} G_k = \begin{bmatrix} n \\ m+1 \end{bmatrix} - \frac{1}{2} n! \sum_{l=0}^{n-1} \left(\frac{1}{2}\right)^l \frac{1}{(n-1-l)!} \begin{bmatrix} n-1-l \\ m+1 \end{bmatrix}. \tag{20}$$

Proof Obviously, $\left(\frac{(m+k)!}{k!} \begin{bmatrix} n \\ m+k \end{bmatrix}\right)$ is the exponential Riordan array $\left(\left(\log \frac{1}{1-t}\right)^m, \frac{1}{t} \log \frac{1}{1-t}\right)_E$. From (8), we have

$$\sum_{k=0}^n \frac{(m+k)!}{k!} \begin{bmatrix} n \\ m+k \end{bmatrix} G_k = \left[\frac{t^n}{n!} \left(\log \frac{1}{1-t} \right)^m \left[\frac{2y}{1+e^y} \mid y = \log \frac{1}{1-t} \right] \right]$$

$$\begin{aligned}
 &= \left[\frac{t^n}{n!} \right] \left(\log \frac{1}{1-t} \right)^m 2 \log \frac{1}{1-t} \cdot \frac{1-t}{2-t} = \left[\frac{t^n}{n!} \right] \left(\log \frac{1}{1-t} \right)^{m+1} \frac{1-t}{1-\frac{t}{2}} \\
 &= \left[\frac{t^n}{n!} \right] \left(\log \frac{1}{1-t} \right)^{m+1} - \frac{n!}{2} [t^{n-1}] \frac{1}{1-\frac{t}{2}} \left(\log \frac{1}{1-t} \right)^{m+1} \\
 &= (m+1)! \left[\begin{matrix} n \\ m+1 \end{matrix} \right] - \frac{n!}{2} \sum_{l=0}^{n-1} \left(\frac{1}{2} \right)^l \frac{(m+1)!}{(n-1-l)!} \left[\begin{matrix} n-1-l \\ m+1 \end{matrix} \right]. \quad \square
 \end{aligned}$$

Similarly, we have

Theorem 6 For $n, m \in \mathbb{N}$, there holds

$$\frac{1}{m+1} \sum_{k=0}^n \binom{m+k}{k} (-1)^{n-m-k} \left[\begin{matrix} n \\ m+k \end{matrix} \right] G_k = n! \sum_{l=0}^n \frac{1}{l!} (-1)^{l-m-1} \left[\begin{matrix} l \\ m+1 \end{matrix} \right] \left(-\frac{1}{2} \right)^{n-l}. \quad (21)$$

Using the exponential Riordan array (12), we can derive the closed form of the following double sum

Theorem 7 For $n \geq 2$, we have

$$\begin{aligned}
 \sum_{l=0}^n \sum_{k=l}^n \binom{n-1}{k-1} \frac{2^k}{k!} (-1)^{k-l} \left[\begin{matrix} k \\ l \end{matrix} \right] G_l &= \frac{(-1)^{n-1} + 1}{n} - \frac{(-1)^{n-2} + 1}{n-1} \\
 &= \begin{cases} \frac{2}{n}, & n \text{ odd,} \\ -\frac{2}{n-1}, & n \text{ even.} \end{cases} \quad (22)
 \end{aligned}$$

Proof It is well known that

$$\frac{n!}{k!} \binom{n-1}{k-1} 2^k = \left[\frac{t^n}{n!} \right] \frac{\left(\frac{2t}{1-t} \right)^k}{k!}.$$

Then $\left(\frac{n!}{k!} \binom{n-1}{k-1} 2^k \right)$ is the exponential Riordan array $\left(1, \frac{2}{1-t} \right)_E$. From (8) and (9), we have

$$\sum_{k=l}^n \binom{n-1}{k-1} 2^k \frac{n!}{k!} (-1)^{k-l} \left[\begin{matrix} k \\ l \end{matrix} \right] = \left[\frac{t^n}{n!} \right] \left[\frac{(\log(1+y))^l}{l!} \Big|_{y = \frac{2t}{1-t}} \right] = \left[\frac{t^n}{n!} \right] \frac{\left(\log \frac{1+t}{1-t} \right)^l}{l!}.$$

Thus $\left(\sum_{k=l}^n \binom{n-1}{k-1} 2^k \frac{n!}{k!} (-1)^{k-l} \left[\begin{matrix} k \\ l \end{matrix} \right] \right)$ is the exponential Riordan array $\left(1, \frac{1}{t} \log \frac{1+t}{1-t} \right)_E$. Moreover, we can get

$$\begin{aligned}
 \frac{1}{n!} \sum_{l=0}^n \sum_{k=l}^n \binom{n-1}{k-1} 2^k \frac{n!}{k!} (-1)^{k-l} \left[\begin{matrix} k \\ l \end{matrix} \right] G_l &= \frac{1}{n!} \left[\frac{t^n}{n!} \right] \left[\frac{2y}{e^y + 1} \Big|_{y = \log \frac{1+t}{1-t}} \right] \\
 &= [t^n] (1-t) \log \frac{1+t}{1-t} = [t^n] \log \frac{1+t}{1-t} - [t^{n-1}] \log \frac{1+t}{1-t} \\
 &= [t^n] (\log(1+t) - \log(1-t)) - [t^{n-1}] (\log(1+t) - \log(1-t)) \\
 &= \frac{(-1)^{n-1}}{n} + \frac{1}{n} - \frac{(-1)^{n-2}}{n-1} - \frac{1}{n-1} = \frac{(-1)^{n-1} + 1}{n} - \frac{(-1)^{n-2} + 1}{n-1}. \quad \square
 \end{aligned}$$

3. Genocchi numbers and Cauchy numbers

According to Comtet^[1], two kinds of Cauchy numbers are defined as the value of a definite integral. The Cauchy numbers of the first type are $\mathcal{C}_n = \int_0^1 (x)_n dx$, where $(x)_n = x(x-1) \cdots (x-n)$

$n + 1$) is the falling factorial, and the Cauchy numbers of the second type are $\hat{C}_n = \int_0^1 \langle x \rangle_n dx$, where $\langle x \rangle_n = x(x + 1) \cdots (x + n - 1)$ is the rising factorial. Merlini et al.^[15] have proved that the generating functions of the Cauchy numbers of the first type and of the second type are

$$\mathcal{E}(\mathcal{C}_n) = \frac{t}{\log(1+t)} \quad \text{and} \quad \mathcal{E}(\hat{\mathcal{C}}_n) = \frac{t}{(1+t)\log(1+t)}, \tag{23}$$

respectively. Moreover, they introduced a definition: if $r \in \mathbb{Z}$, they call Cauchy numbers of the r -th kind the numbers $\mathcal{C}_n^{[r]}$, whose exponential generating function is

$$\mathcal{E}(\mathcal{C}_n^{[r]}) = \frac{t(1+t)^{1-r}}{\log(1+t)}. \tag{24}$$

In this way, we have $\mathcal{E}(\mathcal{C}_n) = \mathcal{E}(\mathcal{C}_n^{[1]})$ and $\mathcal{E}(\hat{\mathcal{C}}_n) = \mathcal{E}(\mathcal{C}_n^{[2]})$ as expected.

By means of the Riordan array method, we can get the closed form or asymptotic value of some double sums related to the Genocchi numbers and the Cauchy numbers.

Theorem 8 For $n \geq 1$, there holds:

$$\sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l = \left(-\frac{1}{2}\right)^{n-1} n!. \tag{25}$$

Proof For

$$\sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} = \left[\frac{t^n}{n!} \right] \frac{t}{\log(1+t)} \frac{(\log(1+t))^l}{l!},$$

$\left(\sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix}\right)$ is the exponential Riordan array $\left(\frac{t}{\log(1+t)}, \frac{1}{t} \log(1+t)\right)_E$. From (8), we have

$$\begin{aligned} \sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l &= \left[\frac{t^n}{n!} \right] \frac{t}{\log(1+t)} \left[\frac{2y}{e^y + 1} \middle| y = \log(1+t) \right] \\ &= \left[\frac{t^n}{n!} \right] \frac{2t}{2+t} = n! [t^{n-1}] \frac{1}{1 + \frac{t}{2}} = \left(-\frac{1}{2}\right)^{n-1} n!. \quad \square \end{aligned}$$

Theorem 9 The following identity holds true

$$\sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \hat{\mathcal{C}}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l = 2n! \left[\left(-\frac{1}{2}\right)^n - (-1)^n \right]. \tag{26}$$

Proof For

$$\sum_{k=l}^n \binom{n}{k} \hat{\mathcal{C}}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} = \left[\frac{t^n}{n!} \right] \frac{t}{(1+t)\log(1+t)} \frac{(\log(1+t))^l}{l!},$$

$\left(\sum_{k=l}^n \binom{n}{k} \hat{\mathcal{C}}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix}\right)$ is the exponential Riordan array $\left(\frac{t}{(1+t)\log(1+t)}, \frac{1}{t} \log(1+t)\right)_E$. From (8), we have

$$\begin{aligned} \sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \hat{\mathcal{C}}_{n-k} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l &= \left[\frac{t^n}{n!} \right] \frac{t}{(1+t)\log(1+t)} \left[\frac{2y}{e^y + 1} \middle| y = \log(1+t) \right] \\ &= \left[\frac{t^n}{n!} \right] \frac{2t}{(1+t)(2+t)} = \left[\frac{t^n}{n!} \right] \left(\frac{4}{2+t} - \frac{2}{1+t} \right) = 2 \left[\frac{t^n}{n!} \right] \frac{1}{1 + \frac{t}{2}} - 2 \left[\frac{t^n}{n!} \right] \frac{1}{1+t} \end{aligned}$$

$$= 2n! \left[\left(-\frac{1}{2} \right)^n - (-1)^n \right]. \quad \square$$

For the Cauchy numbers of the r -th kind, we have the asymptotic value of a double sum related to it.

Theorem 10 For $r \geq 2$, we have

$$\frac{1}{n!} \sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k}^{[r]} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l \sim \frac{2(-1)^{n-1} (n-1)^{r-2}}{(r-2)!}. \quad (27)$$

Proof Because

$$\sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k}^{[r]} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} = \left[\frac{t^n}{n!} \right] \frac{t(1+t)^{1-r} (\log(1+t))^l}{\log(1+t) l!},$$

$\left(\sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k}^{[r]} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} \right)$ is the exponential Riordan array $\left(\frac{t(1+t)^{1-r}}{\log(1+t)}, \frac{1}{t} \log(1+t) \right)_E$. From (8), we have

$$\begin{aligned} \frac{1}{n!} \sum_{l=0}^n \sum_{k=l}^n \binom{n}{k} \mathcal{C}_{n-k}^{[r]} (-1)^{k-l} \begin{bmatrix} k \\ l \end{bmatrix} G_l &= \frac{1}{n!} \left[\frac{t^n}{n!} \right] \frac{2t(1+t)^{1-r}}{2+t} = [t^n] \frac{2t}{(1+t)^{r-1}(2+t)} \\ &= [t^{n-1}] \frac{1}{(1+t)^{r-1} \left(1 + \frac{t}{2}\right)}. \end{aligned}$$

By means of Darboux's method, we know $[t^n] \frac{1}{(1+t)^{r-1} \left(1 + \frac{t}{2}\right)} \sim \frac{2(-1)^n n^{r-2}}{(r-2)!}$. □

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