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Even Signed Permutations Avoiding 2-Letter Signed Patterns

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Abstract Let $\mathcal{D}_n$ be the set of all signed permutations on $[n]=\{1,\ldots,n\}$ with even signs, and let $\mathcal{D}_n(T)$ be the set of all signed permutations in $\mathcal{D}_n$ which avoids a set T of signed patterns. In this paper, we find all the cardinalities of the sets $\mathcal{D}_n(T)$ where $T\subseteq B_2$. Some of the cardinalities encountered involve inverse binomial coefficients, binomial coefficients, Catalan numbers, and Fibonacci numbers.

Keywords forbidden pattern; signed permutation; Catalan number.

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1. Introduction

Let S_n and B_n be the symmetric and hyperoctahedral groups, respectively, on n letters. We regard elements of the hyperoctahedral group B_n as signed permutations written as $\pi = \pi_1\pi_2\ldots\pi_n$ in which each of the symbols $1,2,\ldots,n$ appears, possibly signed (barred). Clearly, the cardinality of B_n is $2^n n!$. We define the signing operation as the one which changes the symbol π_i to $-\pi_i$ and $-\pi_i$ to π_i, so it is an involution, and define the absolute value notation by $|\pi_i|$ to be π_i if π_i unbarred and to be $-\pi_i$ otherwise.

A signed permutation $\pi\in B_n$ is said to contain a pattern $\alpha\in B_k$ if there exists a sequence $1\le i(1)<\cdots<i(k)\le n$ such that

- $\{|\pi_{i(1)}|,\ldots,|\pi_{i(k)}|\}$ is an occurrence of the pattern $\{|\alpha_1|,\ldots,|\alpha_k|\}$, and,
- $\pi_{i(j)}>0$ if and only if $\alpha_j>0$ for all $1\le j\le k$.

A signed permutation π which does not contain such a pattern α is said to avoid α. In this context α is usually called the pattern.

A signed permutation $\pi\in B_n$ is said to be even if the number of barred symbols in π is an even number, that is, the cardinality of the set $\{\pi_i \mid \pi_i=-|\pi_i|\}$ is an even number. Let $\mathcal{D}_n := \{\pi\in B_n \mid \pi \text{ is an even-signed permutation}\}$ be the set of even-signed permutations in B_n. In fact $\mathcal{D}_n$ is a normal subgroup of the hyperoctahedral group B_n. A signed permutation $\pi\in B_n$ is said to be odd if it is not even, and the set of odd signed permutations in B_n is denoted

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by $\mathcal{D}'_n$. Denote by $\mathcal{D}_n(T)$ and $\mathcal{D}'_n(T)$ the collection of even-signed permutations and odd signed permutations which avoid each element in the set T of signed permutations, respectively. Define $d_n(T) = |\mathcal{D}_n(T)|$ and $d'_n(T) = |\mathcal{D}'_n(T)|$, for all $n \geq 0$. Clearly, for all $n \geq 0$ and for any set T of patterns we have

$$|B_n(T)| = d_n(T) + d'_n(T). \tag{1.1}$$

We define three simple operations on signed permutations: the reversal (i.e., reading the permutation right-to-left: $\pi_1\pi_2\cdots\pi_n \mapsto \pi_n\cdots\pi_2\pi_1$), the signing (i.e., $\pi_1\pi_2\cdots\pi_n \mapsto (\text{-}\pi_1)(\text{-}\pi_2)\cdots(\text{-}\pi_n)$), and the complement (i.e., $\pi_1\pi_2\ldots\pi_n \mapsto \beta_1\beta_2\ldots\beta_n$, where $\beta_i = n+1-\pi_i$ if π unbarred, $\beta_i = -(n+1-|\pi_i|)$ if π barred. Let us denote by G_b the group which is generated by the signing operation and the composition of the reversal and the complement operations.

Proposition 1.1 *Let T be any set of patterns. Then $d_n(T) = d_n(g(T))$ and $d'_n(T) = d'_n(g(T))$, where $g(T) = \{g(\alpha) \mid \alpha \in T\}$ and g is either the complement or reversal operation. Moreover, $d_{2n}(T) = d_{2n}(g(T))$ and $d_{2n+1}(T) = d'_{2n+1}(g(T))$ if g is the signed opertaion.*

In the symmetric group S_n, for every 2-letter pattern τ the number of τ-avoiding permutations is 1, and for every pattern $\tau \in S_3$ the number of τ-avoiding permutations is given by the Catalan numbers. Simion[5, Section 3] proved there are similar results for the hyperoctahedral group B_n (generalized by Mansour[2]). For every 2-letter signed pattern τ, the number of τ-avoiding signed permutations is given by $\sum_{j=0}^{n}\binom{n}{j}^2 j!$. Mansour and West[3] enumerated the collections of signed permutations that avoid a signed pattern T, $B_n(T)$, for all possible $T \subseteq B_2$. Recently Dukes and Mansour[1] found the cardinalities of the set of involutions in $B_n(T)$ for all possible $T \subseteq B_2$. In this paper, we find the cardinalities of the set of $\mathcal{D}_n(T)$ for all possible $T \subseteq B_2$ (The exhaustive treatment of cases was suggested by the influential paper of Simion and Schmidt[5], which followed a similar program for the cardinalities $|S_n(T)|$ where $T \subseteq S_3$). The paper is organized as follows. In Sections 2 and 3, we treat the cases $|T| = 1$ and $|T| = 2$, respectively. Finally, in Section 4 we present all the values $d_n(T)$ where $T \subseteq B_2$ such that $|T| \geq 3$.

Pattern τ	Cardinality of $\mathcal{D}_n(\tau)$
$12, 21$	$n!\sum\limits_{j=0}^{[n/2]}\frac{1}{(2[n/2]-2j)!}$
$1\bar{2}, \bar{2}1, \bar{1}2, 2\bar{1}$	$\frac{1}{2}\left(\sum\limits_{j=0}^{n}\binom{n}{j}^2 + \frac{n!}{2^n}\binom{n}{n/2}\right)$
$\overline{12}, \overline{21}$	$n!\sum\limits_{j=0}^{[n/2]}\frac{1}{(n-2j)!}$

Talbe 1 The cardinality of $\mathcal{D}_n(\tau)$ where $\tau \in B_2$.

2. Single 2-letter pattern

In this section we find all the cardinalities $d_n(\tau)$ where $\tau \in B_2$, see Table 1. By taking advantage of Proposition 1.1 together with Equation 1.1, the question of determining the values $d_n(\tau)$ for the 8 choices of one 2-letter signed pattern in B_2, reduces to 4 cases, which are $\tau = 12$, $\tau = \overline{12}$, $\tau = \overline{1}2$, and $\tau = 1\overline{2}$.

2.1 $\tau = 12$ or $\tau = \overline{12}$

In this subsection we find an explicit formula for $d_n(12)$ and $d'_n(12)$.

Theorem 2.1 *For any integer $n \geq 0$, we have*

$$d_{2n}(12) = d_{2n}(\overline{12}) = \sum_{j=0}^{n} \binom{2n}{2j}^2 2j!,\ d_{2n+1}(12) = d'_{2n+1}(\overline{12}) = \sum_{j=0}^{n} \binom{2n+1}{2j+1}^2 (2j+1)!,$$

$$d'_{2n}(12) = d'_{2n}(\overline{12}) = \sum_{j=0}^{n-1} \binom{2n}{2j+1}^2 (2j+1)!,\ d'_{2n+1}(12) = d_{2n+1}(\overline{12}) = \sum_{j=0}^{n} \binom{2n+1}{2j}^2 2j!.$$

Proof We can choose an even-signed permutation in $\mathcal{D}_m(12)$ by choosing $m-j$ unbarred symbols, and $m-j$ positions where $0 \leq j \leq m$ such that j is an even number, and in the other positions we put any permutation with the barred symbols. Hence

$$d_{2n}(12) = d_{2n}(\overline{12}) = \sum_{j=0}^{n} \binom{2n}{2j}^2 2j! \ \text{ and } d_{2n+1}(12) = d'_{2n+1}(\overline{12}) = \sum_{j=0}^{n} \binom{2n+1}{2j+1}^2 (2j+1)!.$$

Using the barred operation, see Proposition 1.1, together with (1.1), we get that $d_{2n}(12) = d_{2n}(\overline{12})$, $d_{2n+1}(12) = d'_{2n+1}(\overline{12})$, $d'_{2n}(12) = d'_{2n}(\overline{12})$, and $d'_{2n+1}(12) = d_{2n+1}(\overline{12})$, as claimed.□

2.2 $\tau = 1\overline{2}$ or $\tau = \overline{1}2$

In this subsection we present explicit formulae for $d_n(1\overline{2})$ and $d'_n(1\overline{2})$.

Lemma 2.2 *For any integer $n \geq 0$, we have*

$$d_n(1\overline{2}) = nd_{n-1}(1\overline{2}) + \sum_{j=2,4,\ldots} \binom{n-1}{j-1}(j-1)!d_{n-j}(1\overline{2}) + \sum_{j=1,3,\ldots} \binom{n-1}{j-1}(j-1)!d'_{n-j}(1\overline{2});$$

$$d'_n(1\overline{2}) = nd'_{n-1}(1\overline{2}) + \sum_{j=2,4,\ldots} \binom{n-1}{j-1}(j-1)!d'_{n-j}(1\overline{2}) + \sum_{j=1,3,\ldots} \binom{n-1}{j-1}(j-1)!d_{n-j}(1\overline{2}).$$

Proof Let $\pi \in \mathcal{D}_n(1\overline{2})$ such that $|\pi_j| = n$. If $\pi_j = n$, then $\pi \in \mathcal{D}_n(1\overline{2})$ if and only if $(\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n) \in \mathcal{D}_{n-1}(1\overline{2})$. So in this case there are $nd_{n-1}(1\overline{2})$ even-signed permutations. Otherwise $\pi_j = \overline{n}$, then all the symbols π_i where $i \leq j-1$ are barred. Hence, $\pi \in \mathcal{D}_n(1\overline{2})$ if and only if $(\pi_{j+1}, \ldots, \pi_n) \in \mathcal{D}_{n-j}(1\overline{2})$ where j is an even number or $(\pi_{j+1}, \ldots, \pi_n) \in \mathcal{D}'_{n-j}(1\overline{2})$ where j is an odd number. So in this case there are

$$\sum_{j=2,4,\ldots} \binom{n-1}{j-1}(j-1)!d_{n-j}(1\overline{2}) + \sum_{j=1,3,\ldots} \binom{n-1}{j-1}(j-1)!d'_{n-j}(1\overline{2})$$

even-signed permutations. Hence, if adding the above two cases, then the formula for $d_n(1\overline{2})$ holds. Similar arguments give the formula for $d'_n(1\overline{2})$. □

Let $D_T(x)$ [resp. $D'_T(x)$] be the exponential generating function for the number of even-signed [resp. odd-signed] permutations in $\mathcal{D}_n(T)$ [resp. $\mathcal{D}'_n(T)$], that is, $D_T(x) = \sum_{n\geq 0} d_n(T)x^n/n!$ [resp. $D'_T(x) = \sum_{n\geq 0} d'_n(T)x^n/n!$]. Using the following lemma which holds immediately by definitions, we obtain explicit formulae for the generating functions $D_{1\overline{2}}(x)$ and $D'_{1\overline{2}}(x)$.

Lemma 2.3 *Let $A(x)$ be the generating function for the sequence $\{a_n\}_{n\geq 0}$. Then*

(1) the generating function for $\{\sum_{j=2,4,\ldots} a_{n-j}\}_{n\geq 0}$ is given by $\frac{x^2A(x)}{1-x^2}$,

(2) the generating function for $\{\sum_{j=1,3,\ldots} a_{n-j}\}_{n\geq 0}$ is given by $\frac{xA(x)}{1-x^2}$.

Lemma 2.2 gives for $n \geq 1$,

$$\begin{cases} \frac{nd_n(1\overline{2})}{n!} = \frac{nd_{n-1}(1\overline{2})}{(n-1)!} + \sum_{j=2,4,\ldots} \frac{d_{n-j}(1\overline{2})}{(n-j)!} + \sum_{j=1,3,\ldots} \frac{d'_{n-j}(1\overline{2})}{(n-j)!}, \\ \frac{nd'_n(1\overline{2})}{n!} = \frac{nd'_{n-1}(1\overline{2})}{(n-1)!} + \sum_{j=2,4,\ldots} \frac{d'_{n-j}(1\overline{2})}{(n-j)!} + \sum_{j=1,3,\ldots} \frac{d_{n-j}(1\overline{2})}{(n-j)!}. \end{cases}$$

If multiplying by x^n and summing over all $n \geq 1$ together with using Lemma 2.3, we get that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}(D_{1\overline{2}}(x) - 1 - xD_{1\overline{2}}(x)) = \frac{1}{1-x^2}(xD_{1\overline{2}}(x) + D'_{1\overline{2}}(x)), \\ \frac{\mathrm{d}}{\mathrm{d}x}(D'_{1\overline{2}}(x) - 1 - xD'_{1\overline{2}}(x)) = \frac{1}{1-x^2}(xD'_{1\overline{2}}(x) + D_{1\overline{2}}(x)). \end{cases}$$

Define $M_T(x) = D_T(x) - D'_T(x)$ for any set T of signed patterns. Subtracting the above two equations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}((1-x)M_{1\overline{2}}(x)) = -\frac{1-x}{1-x^2}M_{1\overline{2}}(x).$$

Clearly, $M_{1\overline{2}}(0) = 1$. Hence, $M_{1\overline{2}}(x) = \frac{1}{\sqrt{1-x^2}}$, which implies the following proposition.

Proposition 2.4 *For any integer $n \geq 0$, we have*

$$d_n(1\overline{2}) - d'_n(1\overline{2}) = \frac{n!}{2^n}\binom{n}{\frac{n}{2}}.$$

Using Proposition 2.4 and the fact that $|B_n(1\overline{2})| = \sum_{j=0}^{n}\binom{n}{j}^2 j!$[4] together with (1.1), we have explicit expressions for $d_n(1\overline{2})$ and $d'_n(1\overline{2})$.

Theorem 2.5 *For any integer $n \geq 0$, we have*

$$d_n(1\overline{2}) = \frac{1}{2}\Big\{\sum_{j=0}^{n}\binom{n}{j}^2 j! + \frac{n!}{2^n}\binom{n}{\frac{n}{2}}\Big\} \text{ and } d'_n(1\overline{2}) = \frac{1}{2}\Big\{\sum_{j=0}^{n}\binom{n}{j}^2 j! - \frac{n!}{2^n}\binom{n}{\frac{n}{2}}\Big\},$$

where $\binom{a}{b}$ is assumed 0 whenever b or a is a non-integer number.

3. Pair 2-letter signed patterns

By taking advantage of Proposition 1.1 together with (1.1), the second question of determining the values $d_n(\tau, \tau')$ for 28 choices of two 2-letter signed patterns, reduces to 8 cases.

3.1 The pair $\{12, \overline{1}2\}$

Theorem 3.1 *For any integer $n \geq 0$, we have that $d_n(12, \overline{1}2) = d'_n(12, \overline{1}2) = (n+1)!$.*

Proof Let $\pi \in \mathcal{D}_n(12, \overline{1}2)$ [resp. $\pi \in \mathcal{D}'_n(12, \overline{1}2)$] such that $|\pi_j| = n$. If $\pi_j = n$, then $j = 1$, so $\pi \in \mathcal{D}_n(12,\overline{1}2)$ [resp. $\pi \in \mathcal{D}'_n(12,\overline{1}2)$] if and only if $(\pi_2,\ldots,\pi_n)$ in $\mathcal{D}_{n-1}(12,\overline{1}2)$ [resp. $\mathcal{D}'_{n-1}(12,\overline{1}2)$]. So in this case there are $d_{n-1}(12,\overline{1}2)$ [resp. $d'_{n-1}(12,\overline{1}2)$] even-signed [resp. odd-signed] permutations. Otherwise $\pi_j = \overline{n}$, it is easy to see that $\pi \in \mathcal{D}_n(12,\overline{1}2)$ [resp. $\pi \in \mathcal{D}'_n(12,\overline{1}2)$] if and only if $(\pi_1,\ldots,\pi_{j-1},\pi_{j+1},\ldots,\pi_n)$ in $\mathcal{D}'_{n-1}(12,\overline{1}2)$ [resp. $\mathcal{D}_{n-1}(12,\overline{1}2)$]. So in this case there are $nd'_{n-1}(12,\overline{1}2)$ [resp. $nd_{n-1}(12,\overline{1}2)$] even-signed [resp. odd-signed] permutations. Hence, if adding the above two cases, we get that for all $n \geq 1$,

$$\begin{cases} d_n(12,\overline{1}2) = d_{n-1}(12,\overline{1}2) + nd'_{n-1}(12,\overline{1}2), \\ d'_n(12,\overline{1}2) = d'_{n-1}(12,\overline{1}2) + nd_{n-1}(12,\overline{1}2). \end{cases}$$

Besides, $d_0(12,\overline{1}2) = d'_0(12,\overline{1}2) = 1$, hence, by induction on n, we get the desired result. □

3.2 The pair $\{12,\overline{12}\}$ or $\{12,\overline{21}\}$

Theorem 3.2 *For any integer $n \geq 0$,*

$$d_{2n+1}(12,\tau) = \frac{1}{2}\binom{4n+2}{2n+1},\ d_{2n}(12,\tau) = \frac{1}{2}\binom{4n}{2n} + \frac{(-1)^n}{2}\binom{2n}{n},$$
$$d'_{2n+1}(12,\tau) = \frac{1}{2}\binom{4n+2}{2n+1},\ d'_{2n}(12,\tau) = \frac{1}{2}\binom{4n}{2n} - \frac{(-1)^n}{2}\binom{2n}{n},$$

where $\tau = \overline{12}$ or $\tau = \overline{21}$.

Proof Let $\pi \in B_n(12,\overline{12})$. Since π avoids 12 [resp. $\overline{12}$], we have the subsequence of all the symbols which are unbarred [resp. barred] in π is decreasing. Let j be the number of the symbols in π which are barred. So

$$d_n(12,\overline{12}) = \sum_{j=0}^{[n/2]} \binom{n}{2j}^2 \text{ and } d'_n(12,\overline{12}) = \sum_{j=0}^{[(n-1)/2]} \binom{n}{2j+1}^2.$$

Hence, the theorem holds for $\tau = \overline{12}$.

Now let us construct a bijection f between the set $\mathcal{D}_n(12,\overline{12})$ and the set $\mathcal{D}_n(12,\overline{21})$ as follows. Let $\pi \in \mathcal{D}_n(12,\overline{12})$ such that $\pi_{i_1},\pi_{i_2},\ldots,\pi_{i_m}$ are all the symbols of π which are barred. We define $f(\pi)$ by π' where $\pi'_j = \pi_j$ if $j \notin \{i_1,i_2,\ldots,i_m\}$, and $\pi'_{i_j} = \pi_{i_{m+1-j}}$ for all $j = 1,2,\ldots,m$. For example, $f(3\overline{12}) = 3\overline{21}$. It is easy to see that f is a bijection, which completes the proof. □

3.3 The pair $\{12,21\}$

Theorem 3.3 *For any integer $n \geq 0$, we have*

$$d_{2n+1}(12,21) = (2n+1)(2n+1)!,\ d_{2n}(12,21) = (2n)!,$$
$$d'_{2n+1}(12,21) = (2n+1)!,\ d'_{2n}(12,21) = 2n(2n)!.$$

Proof Let $\pi \in \mathcal{D}_n(12,21)$. Since π avoids 12 and 21, we have that either π contains exactly one symbol unbarred, or all the symbols of π barred. In the first case, we can choose π by choosing the unbarred symbol, the position of this symbol, and in the other positions we put any order for the other symbols which are barred. So there are $n^2(n-1)!\frac{(1+(-1)^n)}{2}$ even-signed permutations. In the second case, since all the symbols are barred, there are $n!\frac{(1+(-1)^n)}{2}$ even-signed permutations.

Therefore, for all $n \geq 1$, we have

$$d_n(12,21) = n^2(n-1)!\frac{(1+(-1)^{n-1})}{2} + n!\frac{(1+(-1)^n)}{2}.$$

Similar arguments give that for all $n \geq 1$,

$$d'_n(12,21) = n^2(n-1)!\frac{(1-(-1)^{n-1})}{2} + n!\frac{(1-(-1)^n)}{2},$$

as claimed. □

3.4 The pair $\{12, 2\overline{1}\}$

Theorem 3.4 *For any integer $n \geq 1$, we get*

$$d_n(12,2\overline{1}) = n! + n!\sum_{j=0}^{n-2}\frac{1}{(n-j)!}\sum_{d=0}^{[(n-1-j)/2]}\binom{n-1-j}{2d+(j \bmod 2)}(2d+(j \bmod 2))!,$$

$$d'_n(12,2\overline{1}) = n! + n!\sum_{j=1}^{n-2}\frac{1}{(n-j)!}\sum_{d=0}^{[(n-1-j)/2]}\binom{n-1-j}{2d+(j \bmod 2)}(2d+(j \bmod 2))!.$$

Moreover, $M_{12,2\overline{1}}(x) = \frac{1}{x+1}(1+\int_0^x \frac{e^t}{1+t}\mathrm{d}t)$.

Proof Let $\pi \in \mathcal{D}_n(12,2\overline{1})$ such that $|\pi_j| = n$. If $\pi_j = n$, then all the symbols π_i where $i < j$ are barred and all the symbols π_i where $i > j$ are unbarred and decreasing. And since π is even-signed permutations, we get that j is odd number. So, in this case there are $\sum_{j=1}^n \binom{n-1}{j-1}(j-1)!$ even-signed permutation. If $\pi_j = \overline{n}$, then $\pi \in \mathcal{D}_n(12,2\overline{1})$ if and only if $(\pi_1,\ldots,\pi_{j-1},\pi_{j+1},\ldots,\pi_n) \in \mathcal{D}'_{n-1}(12,2\overline{1})$. So in this case there are $nd'_{n-1}(12,2\overline{1})$ even-signed permutations. Therefore, by using the above two cases we get that for all $n \geq 1$,

$$d_n(12,2\overline{1}) = \sum_{j=0,2,4,\ldots}\binom{n-1}{n-1-j}j! + nd'_{n-1}(12,2\overline{1}).$$

Similarly, for all $n \geq 1$, we have

$$d'_n(12,2\overline{1}) = \sum_{j=1,3,\ldots}\binom{n-1}{n-1-j}j! + nd_{n-1}(12,2\overline{1}).$$

Therefore, using induction on n, we obtain the formulae for $d_n(12,2\overline{1})$ and $d'_n(12,2\overline{1})$. Moreover, by Lemma 2.3, we obtain that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}D_{12,2\overline{1}}(x) = \frac{e^x}{1-x^2} + x(xD'_{12,2\overline{1}}(x))', \\ \frac{\mathrm{d}}{\mathrm{d}x}D'_{12,2\overline{1}}(x) = \frac{xe^x}{1-x^2} + x(xD_{12,2\overline{1}}(x))'. \end{cases}$$

Hence $\frac{\mathrm{d}}{\mathrm{d}x}((1+x)M_{12,2\overline{1}}(x)) = \frac{e^x}{1+x}$, which completes the proof. □

3.5 The pair $\{1\overline{2}, \overline{1}2\}$

A split permutation is a permutation $\pi = (\pi', \pi'') \in S_n$, where π' and π'' are nonempty such that every entry of π' is greater than every entry of π''. For example, 231, 312, and 321 are the split permutations in S_3. A d-split permutation is a permutation $\pi = (\pi^1, \pi^2, \ldots, \pi^d)$, where

π^j is nonempty for all j such that every entry of π^j is greater than every entry of π^{j+1} for all $j = 1, 2, \ldots, d-1$. For example, 312 is a 2-split permutation in S_3, namely $\pi^1 = 3$ and $\pi^2 = 12$. Let l_n be the number of all the non-split permutations in S_n, and let $L(x)$ be the corresponding generating function, that is, $L(x) = \sum_{n\geq 0} l_n x^n$.

Lemma 3.5 *We have*
$$L(x) = 2 - \frac{1}{\sum_{k\geq 0} k!x^k}.$$

Proof By definitions, for every permutation $\pi \in S_n$ there exists a unique d such that π is a d-split. Hence, for all $n \geq 0$,
$$n! = \sum_{d\geq 1} \sum_{i_1+i_2+\cdots+i_d=n,\, i_1,\ldots,i_d\geq 1} \prod_{j=1}^{d} l_{i_j}.$$
If multiplying by x^n and summing over all $n \geq 0$, we get
$$\sum_{n\geq 0} n!x^n = 1 + \sum_{d\geq 1} (L(x)-1)^d = \frac{1}{2-L(x)},$$
which is equivalent to $L(x) = 2 - \dfrac{1}{\sum_{k\geq 0} k!x^k}$. □

We say π is a signed-split permutation if there exists π^j such that $\pi = (\pi^1, \pi^2, \ldots, \pi^d)$ where every absolute symbol of π^j is greater than every absolute symbol of π^{j+1}, and either all the symbols of π^j are barred, or unbarred. Clearly, the generating function for number of non-signed-split permutations in B_n is given by $L(2x)$.

Theorem 3.6 *For any integer $n \geq 1$, we have*
$$d_n(1\overline{2}, \overline{1}2) = \sum_{m=1}^{n} \sum_{i_1+\cdots+i_m=n;\ i_j\geq 1} n_{i_1,\ldots,i_m} \prod_{j=1}^{m} l_{i_j},$$
$$d'_n(1\overline{2}, \overline{1}2) = \sum_{m=1}^{n} \sum_{i_1+\cdots+i_m=n;\ i_j\geq 1} (2^m - n_{i_1,\ldots,i_m}) \prod_{j=1}^{m} l_{i_j},$$
where $n_{i_1,\ldots,i_m} = |\{\{b_1,\ldots,b_j\} \subseteq \{i_1,\ldots,i_m\} | (b_1+\cdots+b_j) \bmod 2 = 0\}|$.

Proof Note that π is a signed-split permutation in B_n if and only if $\pi \in B_n(1\overline{2}, \overline{1}2)$ (by induction and considering the maximal s such that $\pi_1, \pi_2, \ldots, \pi_s$ are either unbarred symbols or barred symbols). Therefore, the number of even-signed permutations $\pi \in \mathcal{D}_n(1\overline{2}, \overline{1}2)$ such that $\pi = (\pi^1, \ldots, \pi^m)$, where π^j is a non-split permutation of either barred symbols or unbarred symbols, is given by
$$\sum_{i_1+\cdots+i_m=n;\ i_j\geq 1} n_{i_1,\ldots,i_m} \prod_{j=1}^{m} l_{i_j},$$
and the number of odd signed permutations $\pi \in \mathcal{D}'_n(1\overline{2}, \overline{1}2)$ such that $\pi = (\pi^1, \ldots, \pi^m)$, where π^j is non-split permutation of either barred symbols or unbarred symbols, is given by
$$\sum_{i_1+\cdots+i_m=n;\ i_j\geq 1} (2^m - n_{i_1,\ldots,i_m}) \prod_{j=1}^{m} l_{i_j}.$$

If summing over all possibilities of m, we get the desired result. $\square$

3.6 The pair $\{1\overline{2}, \overline{2}1\}$

Theorem 3.7 *For any integer $n \geq 1$, we have*

$$d_{2n+1}(1\overline{2}, \overline{2}1) = \frac{1}{2}(2n+2)!, \;\; d_{2n}(1\overline{2}, \overline{2}1) = (n+1)(2n)!,$$
$$d'_{2n+1}(1\overline{2}, \overline{2}1) = \frac{1}{2}(2n+2)!, \;\; d'_{2n}(1\overline{2}, \overline{2}1) = n(2n)!.$$

Proof Let $\pi \in \mathcal{D}_n(1\overline{2}, \overline{2}1)$ such that $|\pi_j| = n$. If $\pi_j = n$, then $\pi \in \mathcal{D}_n(1\overline{2}, \overline{2}1)$ if and only if $(\pi_1, \ldots, \pi_{j-1}, \pi_{j+1}, \ldots, \pi_n) \in \mathcal{D}_{n-1}(1\overline{2}, \overline{2}1)$. So in this case there are $nd_{n-1}(1\overline{2}, \overline{2}1)$ even-signed permutations. If $\pi_j = \overline{n}$, then all the symbols of π are barred. So in this case there are $n!\frac{1+(-1)^n}{2}$ even-signed permutations. Therefore, for $n \geq 2$,

$$\begin{cases} d_{2n}(1\overline{2}, \overline{2}1) = (2n)! + 2nd_{2n-1}(1\overline{2}, \overline{2}1), \\ d_{2n+1}(1\overline{2}, \overline{2}1) = (2n+1)d_{2n}(1\overline{2}, \overline{2}1). \end{cases}$$

Similar arguments give for all $n \geq 2$,

$$\begin{cases} d'_{2n+1}(1\overline{2}, \overline{2}1) = (2n+1)! + (2n+1)d'_{2n}(1\overline{2}, \overline{2}1), \\ d'_{2n}(1\overline{2}, \overline{2}1) = 2nd'_{2n-1}(1\overline{2}, \overline{2}1). \end{cases}$$

Besides $d_1(1\overline{2}, \overline{2}1) = 1$ and $d'_1(1\overline{2}, \overline{2}1) = 0$. Hence, by induction on n, we get the desired result.

$\square$

3.7 The pair $\{1\overline{2}, 2\overline{1}\}$

Theorem 3.8 *For any integer $n \geq 1$, we have*

$$d_{2n+1}(1\overline{2}, 2\overline{1}) = \frac{1}{2}(2n+2)!, \;\; d_{2n}(1\overline{2}, 2\overline{1}) = (n+1)(2n)!,$$
$$d'_{2n+1}(1\overline{2}, 2\overline{1}) = \frac{1}{2}(2n+2)!, \;\; d'_{2n}(1\overline{2}, 2\overline{1}) = n(2n)!.$$

Proof Let $\pi \in \mathcal{D}_n(1\overline{2}, 2\overline{1})$. If π_1 is unbarred, then all the symbols of π are unbarred. So there are $n!$ even-signed permutations. If π_1 is barred, then there are $nd'_{n-1}(1\overline{2}, 2\overline{1})$ even-signed permutations. Therefore, by using the above two cases we get that for $n \geq 2$,

$$d_n(1\overline{2}, 2\overline{1}) = n! + nd'_{n-1}(1\overline{2}, 2\overline{1}).$$

Similar arguments give for all $n \geq 2$,

$$d'_n(1\overline{2}, 2\overline{1}) = n! + nd_{n-1}(1\overline{2}, 2\overline{1}).$$

Besides $d_1(1\overline{2}, 2\overline{1}) = d'_1(1\overline{2}, 2\overline{1}) = 1$, hence by induction on n, we get the desired result. $\square$

For $|P|=3$

$d_n(T)$	# sets T
1	2
$1+(n^2-1)e_{n+1}$	2
$2^n-[(n+2)/2]$	4
f_n	8
$\frac{1}{2}(C_{n+1}+(-1)^{n/2}C_{n/2}e_n)$	4
$n!e_n+\sum_{j=1}^{n}\sum_{p+q=n-j\ \ even} p!q!$	2
$\sum_{d=0}^{n}\sum_{i_0+\ldots+i_d=n-d,n-d\ \ even}\prod_{j=0}^{d} i_j!$	2
$n!$	4
$n!+\sum_{j=1}^{[n/2]}\sum_{p+q=n-2j} p!q!$	2
$\sum_{d=0}^{[n/2]}\sum_{i_0+\cdots+i_d=n-2d}\prod_{j=0}^{d} i_j!$	2
$n!\sum_{j=0}^{[n/2]}\frac{1}{(n-2j)!}$	8
$\sum_{j=0}^{[n/2]}(2j)!(n-2j)!$	4
$n!\sum_{j=0}^{[n/2]}\frac{1}{(2j)!}$	8
$n!e_n+n!e_{n+1}\sum_{j=0}^{n}1/j$	4

For $|P|=4$

$d_n(T)$	# sets T
0	1
1	8
n	2
$1+(n-1)e_{n+1}$	4
$1+[n/2]^2+[n/2]e_{n+1}$	2
$1+[n/2]^2+2[n/2]e_{n+1}$	2
$1+\left(\binom{n+1}{2}-1\right)e_{n+1}$	4
2^{n-1}	18
$\sum_{j=0}^{[n/2]}(2j)!$	8
$n!e_n+\sum_{j=0}^{n-1}j!(n-1-j)e_{n+1}$	2
$n!$	11
$\sum_{j=0}^{[n/2]}(n-2j)!$	8

For $|P|=5$

$d_n(T)$	# sets T
0	4
$1+e_{n+1}$	2
1	12
$1+(n-1)e_{n+1}$	10
$[(n+2)/2]$	16
$1+n!e_n$	2
$(2[n/2])!$	4
$n!$	6

For $|P|=6$

$d_n(T)$	# sets T
0	6
1	16
$1+e_n$	4
$n!e_n$	2

For $|P|=7$

$d_n(T)$	# sets T
0	4
e_n	2
1	2

For $|P|=8$

$d_n(T)$	# sets T
0	1

Table 2 The cardinality of $\mathcal{D}_n(P)$ for $|P|\geq 3$ and $n\geq 2$.

4. More than two patterns

Let $P \subseteq B_2$. With the aid of a computer we have calculated the cardinality of $\mathcal{D}_n(P)$ for sets P of three or more patterns. We arrive at these results listed in Table 2, where some are trivially true and some are easy to prove by use of the arguments in the pervious sections. Here, we denote the n-th Fibonacci number ($F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$) by F_n, denote the n-th Catalan number by $C_n = \frac{1}{n+1}\binom{2n}{n}$, and define

$$e_n = (1 + (-1)^n)/2 \quad \text{and} \quad f_n = \begin{cases} \frac{1}{2}(F_{6m} + (-1)^m), & \text{if } n = 3m \\ \frac{1}{2}F_{6m+2}, & \text{if } n = 3m+1 \\ \frac{1}{2}(F_{6m+4} + (-1)^{m+1}), & \text{if } n = 3m+2. \end{cases}$$

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