Journal of Mathematical Research & Exposition
Sept., 2009, Vol. 29, No. 5, pp. 823–830
DOI:10.3770/j.issn:1000-341X.2009.05.007
Http://jmre.dlut.edu.cn

Principally Quasi-Baer Modules

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Abstract In this paper, we give the equivalent characterizations of principally quasi-Baer modules, and show that any direct summand of a principally quasi-Baer module inherits the property and any finite direct sum of mutually subisomorphic principally quasi-Baer modules is also principally quasi-Baer. Moreover, we prove that left principally quasi-Baer rings have Morita invariant property. Connections between Richart modules and principally quasi-Baer modules are investigated.

Keywords principally quasi-Baer rings (modules); endomorphism rings; annihilators; semi-central idempotents.

Document code A
MR(2000) Subject Classification 03C50; 03C95
Chinese Library Classification O153.3

1. Introduction

The concept of principally quasi-Baer rings was first introduced in [1] by Birkenmeier, and further studied by many authors[2−4]. Recall that a ring R is called left (resp. right) principally quasi-Baer (or simply left (resp. right) p.q.-Baer) if the left (resp. right) annihilator of a principal left (resp. right) ideal is generated as a left (resp. right) ideal by an idempotent. This definition is not left-right symmetric. p.q.-Baer rings are the extensions of Baer and quasi-Baer rings[5−11]. The class of p.q.-Baer rings include any domain, any semisimple ring, any Baer and quasi-Baer ring. Our work has been greatly motivated by these works, as mentioned above, and we try to extend these investigations to arbitrary modules.

We define principally quasi-Baer modules on the basis of p.q.-Baer rings. For a left R-module M, we call M a principally quasi-Baer (or simply p.q.-Baer) module if the left annihilator in M of any principal left ideal of S is generated by an idempotent of S. It is easy to see that, when $M = R$, the notion coincides with the existing definition of left p.q.-Baer rings. Thus this definition is not left-right symmetric, either. Among examples of p.q.-Baer modules, we include any semisimple module, any Baer and quasi-Baer module, any finitely generated Abelian ring, any ideal direct summand of a left p.q.-Baer ring (Theorem 2.2), and any finitely generated

Received date: 2007-10-22; **Accepted date**: 2008-03-08
Foundation item: the National Natural Science Foundation of China (No. 10671122).

projective left R-module, where R is a left p.q.-Baer ring (Corollary 2.1). Obviously, any left p.q.-Baer ring R is p.q.-Baer as an R-module.

In Section 2, we introduce the concept of a p.q.-Baer module, and show the equivalent characterizations of p.q.-Baer modules (Theorem 2.1). We prove that any finite direct sum of mutually subisomorphic p.q.-Baer modules is also p.q.-Baer. A natural question arises: for any algebraic property of modules, is the property inherited by direct summands of such a module? We give a positive answer to this question for the case of p.q.-Baer modules (Theorem 2.2). Among other results, we also include results on when direct sums of p.q.-Baer modules are p.q.-Baer (Theorem 2.3) and provide a characterization of p.q.-Baer modules in terms of the FI-strong summand intersection property.

In Section 3, our focus is on the endomorphism rings of p.q.-Baer modules and the connections between p.q.-Baer modules and Richart modules. We show that the endomorphism ring of a p.q.-Baer module is always left p.q.-Baer (Theorem 3.1) and that left p.q.-Baer rings have Morita invariant property. Various conditions on the equivalence of Richart modules and p.q.-Baer modules are discussed.

Throughout this paper, R denotes a ring with unity. For notation we use $S_r(R)$ (resp. $S_l(R)$), $\mathrm{Cen}(R)$, $M_n(R)$ for the right (resp. left) semicentral idempotents of R, the center of R, and the ring of $n \times n$ matrices over R, respectively. M is a left R-module and $S = \mathrm{End}_R(M)$ is the ring of R-endomorphisms of M. Submodules of M will be left R-modules. Recall that a submodule X of M is called fully invariant if for every $h \in S$, $h(X) \subseteq X$. So fully invariant submodules will be an R-S-bimodule. The notations $l_R(\cdot)$ and $r_M(\cdot)$ denote the left annihilator of a subset of M with elements from R and the right annihilator of a subset of R with elements from M, respectively; while $r_S(\cdot)$ and $l_M(\cdot)$ stand for the right annihilator of a subset of M with elements from S and the left annihilator of a subset of S with elements from M, respectively. Let $N \subseteq M$. Then we use $N \leq M$, $N \leq^{\oplus} M$, $N \trianglelefteq M$, $N \trianglelefteq^{\oplus} M$, $N \leq^{e} M$ to denote that N is a submodule, direct summand, fully invariant submodule, fully invariant direct summand, essential submodule of M, respectively.

Before we discuss the properties of p.q.-Baer modules in Section 2, let us recall some related concepts.

Definition 1.1[12] *A left R-module M is called a (quasi-) Baer module if for all $I \leq S_S$ ($I \leq S_S$), $l_M(I) = Me$ where $e^2 = e \in S$.*

Definition 1.2[14] *A ring R is called a left Richart ring if for any element $a \in R$, $l_R(a) = Re$ where $e^2 = e \in R$.*

Definition 1.3[13] *A left R-module M is called a Richart module if for any element $\varphi \in S$, $l_M(\varphi) = Me$ where $e^2 = e \in S$.*

Definition 1.4[2] *An idempotent e of a ring R is called left (resp. right) semicentral if $xe = exe$ (resp. $ex = exe$) for all $x \in R$.*

By [11, Proposition 9] and [1, Example 1.6], we can see that p.q.-Baer rings and Richart rings do not include each other. This is the same as p.q.-Baer modules and Richart modules.

Lemma 1.1[2] *For an idempotent $e \in R$, the following conditions are equivalent:*

(i) $e \in \mathrm{S}_r(R)$;

(ii) $1 - e \in \mathrm{S}_l(R)$;

(iii) Re *is an ideal of* R;

(iv) $(1-e)R$ *is an ideal of* R.

2. Principally quasi-Baer modules

In this section, we begin our investigations by first providing the equivalent characterizations of p.q.-Baer modules and give some properties of them.

Theorem 2.1 *If M is a left R-module, then the following conditions are equivalent:*

(i) M *is p.q.-Baer;*

(ii) The left annihilator in M of every finitely generated left ideal of S is generated by an idempotent of S;

(iii) The left annihilator in M of every principal ideal of S is generated by an idempotent of S;

(iv) The left annihilator in M of every finitely generated ideal of S is generated by an idempotent of S.

Proof We only have to prove (i)⇒(ii) and the rest is clear.

Let $I = \sum_{i=1}^{n} Sx_i$ $(n \in N)$ be any finitely generated left ideal of S. Then $l_M(I) = \bigcap_{i=1}^{n} l_M(Sx_i)$. By hypothesis, we have $l_M(Sx_i) = Me_i$ and $e_i^2 = e_i \in S_r(S)$ $(i = 1, 2 \ldots, n)$. Thus $l_M(I) = \bigcap_{i=1}^{n} Me_i$. Then we assert that $Me_1 \cap Me_2 = Me_1e_2$ and $e_1e_2 \in S_r(S)$.

First let $x \in Me_1 \cap Me_2$. It is easy to check that $x = xe_1 = xe_2 = xe_1e_2 \in Me_1e_2$. Since $e_1 \in S_r(S)$, we have $Me_1e_2 = (Me_1e_2)e_1$ and $Me_1e_2 \subseteq Me_1 \cap Me_2$. It follows that $Me_1e_2 = Me_1 \cap Me_2$. Next, we have $(e_1e_2)^2 = (e_1e_2)e_2 = e_1e_2$, and $e_1e_2x = e_1(e_2x)e_2 = e_1e_2xe_1e_2$ $(\forall x \in S)$ since $e_i \in S_r(S)$ $(i = 1, 2)$. Thus $e_1e_2 \in S_r(S)$.

Similarly, we have $\bigcap_{i=1}^{n} Me_i = M(e_1e_2 \cdots e_n)$ and $(e_1e_2 \cdots e_n) \in S_r(S)$. This completes the proof. □

Theorem 2.2 *Let M be a p.q.-Baer module. Then every direct summand N of M is also a p.q.-Baer module.*

Proof Let $N = Me$ where $e^2 = e \in S$. Then $\mathrm{End}_R(N) = \mathrm{End}_R(Me) \cong eSe$. For any element $x \in \mathrm{End}_R(N)$, we conclude that $l_N(eSe \cdot x) \leq^{\oplus} N$.

First we have $x = exe$, and $y = ye$ for any element $y \in l_N(eSe \cdot x)$. Then $l_N(eSe \cdot x) \subseteq l_M(Sx) \cap N$ since $0 = y \cdot Sx = ye \cdot S \cdot exe = y(eSe)x = 0$. Secondly, let $z \in l_M(Sx) \cap N$. We have $z \in l_N(eSe \cdot x)$ since $z = ze \in N$ and $z \cdot eSe \cdot x = (ze)S(exe) = z \cdot Sx = 0$. This implies $l_N(eSe \cdot x) = l_M(Sx) \cap N$.

By assumption, we have $l_M(Sx) = Mf$ where $f^2 = f \in S_r(S)$. Then $l_M(Sx) \cap N =$

$Mf \cap Me = Me(efe)$, and efe is an idempotent of eSe since $f^2 = f \in S_r(S)$. Therefore, $l_N(eSe \cdot x) = Me(efe) \leq^{\oplus} Me$. □

Example 2.1 Let R be a left p.q.-Baer ring and let $e^2 = e \in R$ be any idempotent of R. Then $M = Re$ is a left R-module which is p.q.-Baer.

Theorem 2.3 *If M_1 and M_2 are p.q.-Baer modules, and have the property that for any $\psi \in \mathrm{Hom}_R(M_i, M_j)$, $\psi(x) = 0$ implies $x = 0$ ($i \neq j, i, j = 1, 2$). Then $M_1 \oplus M_2$ is a p.q.-Baer module.*

Proof Let $S = \mathrm{End}_R(M_1 \oplus M_2)$ and I be any finitely generated ideal of S. By [12, Lemma 1.10], we have $l_{M_1 \oplus M_2}(I) \lhd M_1 \oplus M_2$, and there exists $N_i \lhd M_i$ ($i = 1, 2$) such that $l_{M_1 \oplus M_2}(I) = N_1 \oplus N_2$, where $N_i = l_{M_1 \oplus M_2}(I) \cap M_i$ ($i = 1, 2$).

As mentioned, $S = S_1 \oplus \mathrm{Hom}_R(M_1, M_2) \oplus \mathrm{Hom}_R(M_2, M_1) \oplus S_2$, where $S_i = \mathrm{End}_R(M_i)$ ($i = 1, 2$). Since I is a finitely generated ideal of S, we have $I = I_1 \oplus I_{12} \oplus I_{21} \oplus I_2$, where $I_1 \lhd S_1$, $I_2 \lhd S_2$, $I_{12} = \{\varphi \in \mathrm{Hom}_R(M_2, M_1) | \varphi = \xi_{12}$ with $(\xi_{ij})_{i,j=1,2} \in I\}$, $I_{21} = \{\varphi \in \mathrm{Hom}_R(M_1, M_2) | \varphi = \xi_{21}$ with $(\xi_{ij})_{i,j=1,2} \in I\}$. It is easy to see that I_i is a finitely generated ideal of S_i ($i = 1, 2$).

Let us define $l_{M_i}(I_i) = N_i'$ ($i = 1, 2$). It is easy to check that $N_1 = N_1' \cap (\bigcap_{\varphi \in I_{21}} \ker \varphi)$. Then we conclude that $N_1 = N_1'$. For any element $\psi_{12} \in \mathrm{Hom}_R(M_2, M_1)$, $\varphi \in I_{21}$, we have $N_1'(\varphi\psi_{12}) = 0$. Thus $N_1'\varphi = 0 \Rightarrow N_1' \subseteq \bigcap_{\varphi \in I_{21}} \ker \varphi$. It follows that $N_1 = N_1'$. Similarly, we have $N_2 = N_2'$. Since M_1, M_2 are p.q.-Baer modules and I_i is a finitely generated ideal of S_i, we have $N_i' = l_{M_i}(I_i) \leq^{\oplus} M_i$ ($i = 1, 2$). Therefore $l_{M_1 \oplus M_2}(I) = N_1' \oplus N_2' \leq^{\oplus} M_1 \oplus M_2$. This completes the proof. □

The proof of Theorem 2.3 is similar to [12, Theorem 3.18]. For the completion of this paper, we write down the whole process.

By Theorems 2.2 and 2.3, we have the following result, which provides another source of examples for p.q.-Baer modules.

Proposition 2.1 *Let $M = \bigoplus_{i=1}^n M_i$. If M_i is subisomorphic to (i,e., isomorphic to a submodule of) M_j, $\forall i \neq j$; $i, j = 1, 2, \ldots, n$. Then M is p.q.-Baer if and only if M_i is p.q.-Baer ($i = 1, 2, \ldots, n$).*

It is easy to see that Proposition 2.1 also holds true when $M = \prod_{i=1}^n M_i$. From Proposition 2.1 and Theorem 2.2, we have

Corollary 2.1 *A finitely generated projective module over a left p.q.-Baer ring is a p.q.-Baer module.*

We know that Baer and quasi-Baer modules are p.q.-Baer modules. A natural question arises, is the p.q.-Baer module also a Baer or a quasi-Baer module? The $n \times n$ ($n > 1$) upper triangular matrix ring over a domain, which is not a division ring, is a p.q.-Baer ring but not Baer[3, p16]. Let $R = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} M_{n'}(W) | a_n$ is eventually constant$\}$, where W is the Kth ($K > 1$) Weyl algebra over a field of characteristic Zero[1, Example 3.13]. Then R is p.q.-Baer but not quasi-Baer. So p.q.-Baer modules might be neither Baer nor quasi-Baer. We will ask: under what conditions might p.q.-Baer modules and quasi-Baer modules be equivalent? The following

Proposition answers this question. We define the FI-(strong) summand intersection property on the basis of (strong) summand intersection property[12].

Definition 2.2 *A module M is said to have the FI-summand intersection property (FI-SIP) if the intersection of two fully invariant direct summands is again a direct summand. M has the FI-strong summand intersection property (FI-SSIP) if the intersection of any number of fully invariant direct summands is again a direct summand.*

Proposition 2.2 *A module M is quasi-Baer if and only if M is p.q.-Baer and has the FI-strong summand intersection property (FI-SSIP).*

Proof The first assertion of the necessary condition is clear.

For the second, let $Me_i \lhd M$, $e_i^2 = e_i \in S$, $i \in \Lambda$ (Λ is an index set). Then $e_i \in S_r(S)$, $(1-e_i)S \lhd S$ $(i \in \Lambda)$. Let us define $I = \sum_{i\in\Lambda}(1-e_i)S$. Then $I \lhd S$ and $l_M(I) = \bigcap_{i\in\Lambda} l_M[(1-e_i)S] = \bigcap_{i\in\Lambda} Me_i \leq^{\oplus} M$. Thus, M satisfies the FI-SSIP.

Conversely, let I be any ideal of S. Then we can write $I = \sum_{i\in\Lambda} Sx_iS$ $(x_i \in I, i \in \Lambda)$. So $l_M(I) = l_M(\sum_{i\in\Lambda} Sx_iS) = \bigcap_{i\in\Lambda} l_M(Sx_iS)$. Since M is p.q.-Baer, we have $l_M(Sx_iS) = Me_i \lhd^{\oplus} M$ where $e_i^2 = e_i \in S_r(S)$ $(\forall i \in \Lambda)$. By assumption, $l_M(I) = \bigcap_{i\in\Lambda} Me_i = Me \leq^{\oplus} M$. Hence M is quasi-Baer. □

Recall from [12] that a module M is called $\mathcal{K}$-nonsingular if, for all $\varphi \in S$, $l_M(\varphi) = \ker\varphi \leq^e M$ implies $\varphi = 0$.

By [12, Lemma 2.15] and [13, Theorem 2.4], we know that both Baer and Richart modules are $\mathcal{K}$-nonsingular. The following theorem shows that under a certain condition, a p.q.-Baer module is also $\mathcal{K}$-nonsingular.

Proposition 2.3 *Let M be a p.q.-Baer module. If every essential submodule of M is an essential extension of a fully invariant submodule of M, then M is $\mathcal{K}$-nonsingular.*

Proof Let $0 \neq \varphi \in S$ and $l_M(\varphi) = \ker\varphi \leq^e M$. By hypothesis, there exists a fully invariant submodule $N \lhd M$ such that $N \leq^e l_M(\varphi)$. Then $N \subseteq l_M(S\varphi) = Me$ $(e^2 = e \in S)$ since $NS\varphi = N\varphi = 0$ and M is p.q.-Baer. It follows that $Me \leq^e M$. This implies that $e = 1$, $\varphi = 0$, contradicting our assumption that $\varphi \neq 0$. Thus M is $\mathcal{K}$-nonsingular. □

3. Endomorphism rings, connections between p.q.-Baer and Richart modules

In [12, 13] we can see that the endomorphism rings of any Baer, quasi-Baer and Richart modules are Baer, quasi-Baer and left Richart rings, respectively. This suggests that these modules property may be carried over to their endomorphism rings. In this section, we study the endomorphism rings of p.q.-Baer modules and the connections between p.q.-Baer modules and Richart modules.

Theorem 3.1 *If M is a p.q.-Baer module with $S = \mathrm{End}_R(M)$. Then S is a left p.q.-Baer ring.*

Proof Let I be any principal left ideal of S. We have $l_M(I) = Me$ where $e^2 = e \in S$. Then we conclude that $l_S(I) = Se$.

First, $Se \subseteq l_S(I)$ since $MSeI = MeI = 0$. Next, for any $0 \neq \varphi \in l_S(I)$, we have $M\varphi \subseteq l_M(I)$. Thus $\varphi = \varphi e$. This implies that $l_S(I) \subseteq Se \Rightarrow l_S(I) = Se$. This completes the proof. □

Corollary 3.1 *Let R be a left p.q.-Baer ring and e is an idempotent of R. Then eRe is also a left p.q.-Baer ring.*

Theorem 3.2 *The left p.q.-Baer condition is a Morita invariant property.*

Proof Let R be a left p.q.-Baer ring. By Proposition 2.1, we have $R^{(n)}$ is left p.q.-Baer. Since $M_n(R) \cong \mathrm{End}_R(R^{(n)})$, we know that $M_n(R)$ is also left p.q.-Baer. □

Proposition 3.1 *Let R be a commutative ring. Then the following conditions are equivalent:*

(i) R is left p.q.-Baer;

(ii) R is left Richart;

(iii) R is VN-regular.

Proof It is easy to see that when R is commutative, left p.q.-Baer rings and left Richart rings are equivalent, and the rest is immediate from [13, Theorem 3.2]. □

Corollary 3.2 *Let M be a left p.q.-Baer module. Then* $\mathrm{Cen}(S)$ *is VN-regular.*

Definition 3.1[13] *A module M is called quasi-retractable if* $\mathrm{Hom}_R(M, N) \neq 0$, *where $N = Rm$, $\forall 0 \neq m \in M$ (or, equivalently, $\exists 0 \neq \varphi \in S$ with $M\varphi \subseteq N = Rm$).*

Proposition 3.2 *Let M be quasi-retractable. Then M is p.q.-Baer if and only if S is a left p.q.-Baer ring.*

Proof We only have to prove the sufficient condition. Let I be any principal left ideal of S. we assert that $l_M(I) = Me$.

First, by assumption, we have $l_S(I) = Se$ where $e^2 = e \in S$. Thus $Me \subseteq l_M(I)$ since $MeI \subseteq MSeI = 0$. Next, if $\exists 0 \neq m \in l_M(I) \setminus Me$, by quasi-retractability, there exists $0 \neq \beta \in S$ such that $M\beta \subseteq Rm$. It follows that $\beta = \beta(1 - e) \in S(1 - e)$. Also, we have $\beta \in l_S(I) = Se$ since $M\beta I \subseteq RmI = 0$. This implies that $\beta = 0$, a contradiction. Therefore, $l_M(I) = Me$. □

In the rest, we will consider the connections between p.q.-Baer modules and Richart modules. Similarly to the definitions of the insertion of factors property (IFP)[16] and strongly bounded property [1] of rings, we give the following definitions.

Definition 3.2 *A left R-module M is said to satisfy the IFP (insertion of factors property) if $l_M(\varphi)$ is a fully invariant submodule of M for all $\varphi \in S$ (or, equivalently, $r_s(m) \lhd S$ for all $m \in M$).*

Definition 3.3 *A left R-module M is strongly bounded if every nonzero submodule of M contains a nonzero fully invariant submodule.*

Proposition 3.3 *Let M be p.q.-Baer and strongly bounded. Then M is Richart and satisfies the IFP.*

Proof Let $\varphi \in S$. We have $Me = l_M(S\varphi S) \subseteq l_M(\varphi)$ $(e^2 = e \in S)$. Hence, $l_M(\varphi) = Me \oplus A$ for some $A \leq M$. If $A \neq 0$, by assumption, there exists a fully invariant submodule $0 \neq B \subseteq A$. Then, $B \subseteq l_M(\varphi) \Rightarrow BS \subseteq l_M(\varphi) \Rightarrow BS\varphi = 0 \Rightarrow BS\varphi S = 0$. Thus $B \subseteq Me$, this is impossible. Therefore, $l_M(\varphi) = l_M(S\varphi S) \lhd^{\oplus} M$. M is Richart and satisfies the IFP. □

Proposition 3.4 *Let M be a left R-module that satisfies the IFP. Then*

(i) M is Richart if and only if M is p.q.-Baer;

(ii) S is Abelian.

Proof (i) First, for any $\varphi \in S$, we have $l_M(S\varphi) \subseteq l_M(\varphi)$. Next, for any element $m \in l_M(\varphi)$, we have $\varphi \in r_S(m)$. It follows that $m \in l_M(S\varphi)$ since $r_S(m) \lhd M$ and $S\varphi \subseteq r_S(m)$. Thus $l_M(S\varphi) = l_M(\varphi)$, Richart and p.q.-Baer modules are equivalent;

(ii) The proof is routine. □

Theorem 3.3 *Let M be a left R-module, $S = \mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(i) M is a Richart modules and S is Ablian;

(ii) M is a p.q.-Baer module which satisfies the IFP.

Proof (i)⇒(ii). First, for any $\varphi \in S$, we have $l_M(S\varphi) \subseteq l_M(\varphi)$ and $l_M(\varphi) = Me$ $(e^2 = e \in \mathrm{Cen}(S))$. Then, $eS\varphi = 0$ since $eS\varphi = Se\varphi$ and $e\varphi \subseteq Me\varphi = 0$. It follows that $Me \subseteq l_M(S\varphi)$. Thus $l_M(S\varphi) = Me$. Since S is Ablian, we have $l_M(S\varphi) = Me \lhd M$;

(ii)⇒(i). This is immediate from Proposition 3.4. □

Proposition 3.5 *Let M be a left R-module, $S = \mathrm{End}_R(M)$. Consider the following conditions:*

(a) M satisfies the IFP;

(b) S is reduced;

(c) S satisfies the IFP;

(d) S is Ablian.

The following statements hold true:

(i) If S is a left Richart ring, then (b) through (d) are equivalent;

(ii) If M is a Richart module, then (a) through (d) are equivalent;

(iii) If S is a VN-regular ring, then (a) through (d) are equivalent.

Proof (i) For any ring S, it is easy to get (b)⇒(c)⇒(d). Now, we only have to prove (d)⇒(b). Let $x^2 = 0$. Then $r_R(x) = eS$ where $e^2 = e \in \mathrm{Cen}(S)$. Thus $x = ex = xe = 0$ since $x \in r_R(x) = eS$;

(ii) By [13, Theorem 3.1], we know that S is left Richart. Thus, we only have to prove that (a)⇔(d). By Proposition 3.4 and Theorem 3.3, we know that (a)⇔(d);

(iii) We only have to prove that if S is VN-regular, then M is Richart.

For any $\varphi \in S$, there exists $\psi \in S$ such that $\varphi = \varphi\psi\varphi$. Let us define $\pi = \varphi\psi \in S$. Then $\pi^2 = \pi$ and $\varphi = \pi\varphi$. This implies that $\ker\varphi = \ker\pi = M(1-\pi) \leq^{\oplus} M$. □

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