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Existence Result for Discrete Problems with Dependence on the First Order Difference

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Abstract In this paper, we prove the existence of a positive solution and a negative solution for a class of second order difference equations with dependence on the first order difference. Our proofs are based on the Mountain Pass Lemma and iterative methods.

Keywords existence result; the first order difference; Mountain Pass Lemma; iterative technique.

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1. Introduction

In this paper, we consider the following discrete boundary value problems

$$\text{(P)}\quad \begin{cases} \Delta^2 y(k-1) + f(k, y(k), \Delta y(k)) = 0, & k \in [1, T] \\ y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where $T \in \{1, 2, \ldots\}$, $[1, T]$ is the discrete interval $\{1, 2, \ldots, T\}$, $\Delta y(k) = y(k+1) - y(k)$ is the forward difference operator, $\Delta^2 y(k) = \Delta(\Delta y(k))$, and the function $f : [1, T] \times R \times R \to R$ is locally Lipschitz continuous.

Recently, some authors have studied discrete boundary value problems by using variational methods. In 2003, Guo and Yu[4] investigated the following second order difference equation

$$\Delta^2 y(k-1) + f(k, y(k)) = 0, \tag{1.2}$$

where $f : Z \times R \longrightarrow R$ is a continuous function in the second variable and $f(k+m, z) = f(k, z)$ for all positive integer m and $(k, z) \in Z \times R$. They obtained some multiplicity results for the problem (1.2). In 2004, Agarwal, Perera and O' Regan[1] discussed the singular and nonsingular second order difference equations

$$\begin{cases} \Delta^2 y(k-1) + f(k, y(k)) = 0, & k \in [1, T] \\ y(0) = y(T+1) = 0, \end{cases} \tag{1.3}$$

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where $f \in C([1,T]\times[0,\infty),R)$ satisfies $f(k,0)\geqslant 0$, for all $k\in[1,T]$. Using the minimax principle and Mountain Pass Lemma in critical point theory, they established the existence of multiple positive solutions for the problem (1.3). In 2004, using Mountain Pass Lemma, Zhang and Yang[6] proved the existence of 2^n nontrivial solutions for discrete two-point boundary value problems which come from the steady-state temperature distribution of heat diffusion on multi-body. In 2005, Agarwal, Perera and O' Regan[2] considered the following singular boundary value problem

$$\begin{cases} \Delta(\phi_p(\Delta y(k-1)) + f(k,y(k)) = 0, & k\in[1,T], \\ y(T+1)=0, & k\in[1,T], \\ y(0)=y(T+1)=0, \end{cases} \tag{1.4}$$

where $\phi_p(s)=|s|^{p-2}s$, $1<p<\infty$ and $f\in C([1,T]\times[0,\infty),R)$ satisfies

$$b_0(k)\leqslant f(k,t)\leqslant b_1(k)t^{-\theta}, \quad (k,t)\in[1,T]\times(0,t_0), \tag{1.5}$$

for some nontrivial functions b_0, $b_1\geqslant 0$ and θ, $t_0>0$. It follows that f may be singular at $t=0$ and may change sign. They obtained the existence of multiple positive solutions for the problem (1.4) by using variational methods. In 2006, Zhang and Liu[7] discussed the second order superlinear difference systems

$$\begin{cases} \Delta^2 x(k-1)) + g(k,y(k)) = 0, & k\in[1,T] \\ \Delta^2 y(k-1)) + f(k,x(k)) = 0, & k\in[1,T] \\ x(0)=x(T+1)=y(0)=y(T+1)=0. \end{cases} \tag{1.6}$$

Using minimax principle and Linking Theorem, they proved the existence of two positive solutions for the problem (1.6). In 2007, Zhang, Zhang and Liu[8] investigated a class of second order difference equations with discontinuous nonlinearities, and obtained a new multiplicity result by using a three critical points theorem. However, there are a few approaches to study discrete boundary value problems with dependence on the first order difference. The obvious reason is that, contrary to the problems (1.2), (1.3) and (1.4), Problem (P) does not have variational structure, due to the presence of the first order difference in the nonlinear term. So the critical point theory cannot be directly used to attack Problem (P).

In this paper, we use variational type methods to handle problem (P) under suitable assumptions on the date. Firstly, we “freeze” the first order difference for the function to consider a class of second order difference equations which are independent of the first order difference associated with Problem (P), and thus obtain the existence of a nontrivial solution by using Mountain Pass Lemma. Secondly, using some estimates and an iterative scheme, we obtain the existence of a positive solution and a negative solution for Problem (P).

The paper is arranged as follows. In Section 2, we recall some basic lemmas and state our main theorem; In Section 3, we prove our main theorem.

2. Some basic lemmas and main theorem

In this section, we recall some lemmas and state our main theorem.

Definition 2.1 *Let X be a real Hilbert space, $I \in C^1(X, R)$ which means that I is a continuously Frechet differentiable functional defined on X. I is said to satisfy (PS) condition, if any sequence $(u_n) \subset X$ with*

$$I(u_n) \leq c, (c \in R), \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty \tag{2.1}$$

possesses a convergent subsequence in X.

Lemma 2.2 (Mountain Pass Lemma)[5] *Let X be a real Hilbert space with norm $\| \ \|_X$, the functional $I \in C^1(X, R)$ satisfies (PS) condition $I(0) = 0$ and Assume*

i) There exist $\rho > 0$, $\alpha > 0$ and $\|u\|_X = \rho$, such that $I(u) \geqslant \alpha$;

ii) There exists $u_1 \in X$ and $\|u_1\|_X \geqslant \rho$, such that $I(u_1) \leqslant \alpha$.

Define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1]; X) | \gamma(0) = 0, \gamma(1) = u_1\}$. Then the functional I has a positive critical value $c \geqslant \alpha > 0$.

Now, we define the class H of the functions $y : [0, T + 1] \to R$ with $y(0) = y(T + 1) = 0$. H is a T-dimensional Hilbert space with inner product

$$(y, w) = \sum_{k=1}^{T} (y(k)w(k)), \quad \forall y, w \in H. \tag{2.2}$$

We denote the induced norm by

$$\|y\| = \Big(\sum_{k=1}^{T} y^2(k) \Big)^{\frac{1}{2}}, \quad \forall y \in H. \tag{2.3}$$

It is obvious that $(H, (\ , \ ))$ is linearly homeomorphic to R^T.

Consider the following linear eigenvalue problem

$$\Delta^2 y(k - 1) + \lambda y(k) = 0, \quad y \in H. \tag{2.4}$$

Define the functional Φ on H as follows,

$$\Phi(y) = \frac{1}{2} \sum_{k=1}^{T+1} (\Delta y(k - 1))^2 - \lambda y^{\mathrm{T}} y = \frac{1}{2} y^{\mathrm{T}} A y - \lambda y^{\mathrm{T}} y,$$

where $y = (y(1), y(2), \ldots, y(T))^{\mathrm{T}}$, and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{T \times T}. \tag{2.5}$$

It is obvious that the functional $\Phi \in C^1(H, R)$ and

$$(\Phi'(y), z) = \sum_{k=1}^{T+1} [\Delta y(k - 1)\Delta z(k - 1) - \lambda y(k)z(k)] = -\sum_{k=1}^{T} [\Delta^2 y(k - 1) + \lambda y(k)]z(k), \quad \forall z \in H.$$

So the solutions of the problem (3.4) are precisely the critical points of Φ. The nontrivial solutions of the problem (3.4) can only be found when λ is one of the eigenvalues[3]

$$\lambda_k = 4\sin^2\frac{k\pi}{2(T+1)}, \quad k = 1, 2, \ldots, T, \tag{2.6}$$

of the positive definite matrix A, and the corresponding eigenvector is

$$\xi^k = \sqrt{\frac{2}{(T+1)}}\Big(\sin\frac{k\pi}{(T+1)}, \sin\frac{2k\pi}{(T+1)}, \ldots, \sin\frac{Tk\pi}{(T+1)}\Big)^{\mathrm{T}}, \quad k = 1, 2, \ldots, T.$$

Let $\lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_T\}$, $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_T\}$, where$\lambda_1, \lambda_2, \ldots, \lambda_T$ are defined by (2.6).

Now, we state our main theorem. Assume

(H1) $\lim_{t\to 0}\frac{f(k,t,\xi)}{t} = 0$, uniformly with respect to $k \in [1, T]$, $\xi \in R$;

(H2) There exist $R > 0$ and $\beta > 2$ such that

$$0 < \beta F(k, t, \xi) \leqslant tf(k, t, \xi), \quad \forall |t| \geqslant R, \xi \in R,$$

where $F(k, t, \xi) = \int_0^t f(k, v, \xi)\mathrm{d}v$;

(H3) There exist $a_1 > 0$ and $a_2 > 0$ such that

$$F(k, t, \xi) \geqslant a_1|t|^\beta - a_2;$$

(H4) There exist $p > 1$ and $a_3 > 0$ such that

$$f(k, t, \xi) \leqslant a_3(1 + |t|^p);$$

(H5) The function f satisfies the following local Lipschitz conditions

$$|f(x, t', \xi) - f(x, t'', \xi)| \leqslant L_1|t' - t''|, \quad \forall k \in [1, T]t', t'' \in [0, \rho_1], |\xi| \leqslant \rho_2,$$

$$|f(x, t, \xi') - f(x, t, \xi'')| \leqslant L_2|\xi' - \xi''|, \quad \forall k \in [1, T], t \in [0, \rho_1], |\xi'| \leqslant \rho_2, |\xi''| \leqslant \rho_2,$$

where ρ_1, $\rho_2 > 0$.

Theorem 2.3 *Assume (H1)–(H5) hold. Then Problem (P) has a positive solution and a negative solution, provided*

$$\frac{L_2}{2}\lambda_{\max} + (L_1 + \frac{L_2}{2}) < \lambda_{\min}.$$

3. Proof of Theorem 2.3

For each $w \in H$, we consider the following problem

$$\begin{cases} \Delta^2 y(k-1) + f(k, y(k), \Delta w(k)) = 0, \quad k \in [1, T] \\ y(0) = y(T+1) = 0. \end{cases} \tag{3.1}$$

Now the problem (3.1) is variational, and we can treat it by using variational methods.

Define the functional

$$I_w(y) = \frac{1}{2}\sum_{k=1}^{T+1}(\Delta y(k-1))^2 - \sum_{k=1}^{T}F(k, y, \Delta w(k)), \tag{3.2}$$

for each $w \in H$. From (H1) and (H3), we obtain that the functional $I \in C^1(H, R)$, and by a simple computation we have

$$\begin{aligned}\frac{\partial I_w(y)}{\partial y(k)} &= 2y(k) - y(k+1) - y(k-1) - f(k, y(k), \Delta w(k)) \\ &= -(\Delta^2 y(k-1) + f(k, y(k), \Delta w(k))).\end{aligned}$$

Therefore, a solution of the problem (3.1) is obtained as a critical point of the functional $I_w(y)$.

Lemma 3.1 *Assume (H1)–(H4) hold. Then the problem (3.1) has at least one solution y_w for any $w \in H$, and there exist $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \leqslant \|y_w\| \leqslant c_2.$$

Moreover, under the above hypotheses, the problem (3.1) has a positive solution and a negative solution.

Proof Firstly, we prove the problem (3.1) has at least one nontrivial solution by using Lemma 2.2 (Mountain Pass Lemma).

Claim 1 The functional $I_w(y)$ satisfies (PS) condition. Indeed, let the sequence $\{y_m\} \subset H$ satisfy

$$I_w(y_m) \leqslant c,\ I'_w(y_m) \to 0 \text{ as } m \to \infty. \tag{3.3}$$

Since H is finite dimensional, it suffice to show that the sequence $\{y_m\}$ is bounded. By (3.3), we have

$$\begin{aligned}&\beta c + o(1)\|y_m\| \geq \beta I_w(y_m) - (I'_w(y_m), y_m) \\ &= \frac{\beta-2}{2}\sum_{k=1}^{T+1}(\Delta y_m(k-1))^2 + \sum_{k=1}^{T}(f(k, y_m(k), \Delta w(k))y_m(k) - \beta F(k, y_m(k), \Delta w(k))) \\ &\geq \frac{\beta-2}{2}\|y_m\|^2 + \sum_{N_1}(f(k, y_m(k), \Delta w(k))y_m(k) - \beta F(k, y_m(k), \Delta w(k))) + \\ &\quad \sum_{N_2}(f(k, y_m(k), \Delta w(k))y_m(k) - \beta F(k, y_m(k), \Delta w(k))),\end{aligned} \tag{3.4}$$

where $N_1 = \{k | k \in [1, T], y(k) \geqslant R\}$, $N_2 = [1, T] \backslash N_1$. By (H2), we have

$$\sum_{N_1}(f(k, y_m(k), \Delta w(k))y_m(k) - \beta F(k, y_m(k), \Delta w(k))) \geqslant 0.$$

Therefore, we have

$$\beta c + o(1)\|y_m\| \geqslant \frac{\beta-2}{2}\|y_m\|^2 + \sum_{N_2}(f(k, y_m(k), \Delta w(k))y_m(k) - \beta F(k, y_m(k), \Delta w(k))). \tag{3.5}$$

By (H4), $N_2 = [1, T] \backslash N_1$ and the function $f : [1, T] \times R \times R$ is locally Lipschitz continuous. We obtain that the last term of (3.5) is finite, and hence the sequence $\{y_m\}$ is bounded.

Claim 2 The functional $I_w(y)$ has the geometry of Mountain Pass Lemma (i.e., i), ii) of Lemma

2.2). Indeed by (H1) we have

$$\lim_{t\to 0}\frac{F(k,t,\xi)}{t^2}=0.$$

Let $\varepsilon=\frac{1}{4}\lambda_{\min}>0$. There exists $\rho>0$, such that for t with $|t|=\rho$,

$$F(k,t,\xi)\leqslant\frac{1}{4}\lambda_{\min}t^2.$$

Therefore, we obtain

$$\begin{aligned}I_w(y)&=\frac{1}{2}\sum_{k=1}^{T+1}(\Delta y(k-1))^2-\sum_{k=1}^{T}F(k,y,\Delta w(k))\\&\geq\frac{1}{2}\lambda_{\min}\|y\|^2-\frac{1}{4}\lambda_{\min}\|y\|^2=\frac{1}{4}\lambda_{\min}\|y\|^2.\end{aligned}$$

Let $\alpha=\frac{1}{4}\lambda_{\min}\rho^2$. Then we have

$$I_w(y)\geqslant\alpha,\quad\forall\|y\|=\rho.$$

On the other hand, we fix $y_0\in H$ with $\|y_0\|=1$, and by (H3), we have

$$\begin{aligned}I_w(sy_0)&=\frac{1}{2}s^2\sum_{k=1}^{T+1}(\Delta y_0(k-1))^2-\sum_{k=1}^{T}F(k,sy_0,\Delta w(k))\\&\leq\frac{1}{2}\lambda_{\max}s^2\|y_0\|^2-a_1|s|^\beta\sum_{k=1}^{T}|y_0|^\beta-a_2(T-1)\\&\leq\frac{1}{2}\lambda_{\max}s^2-a_1|s|^\beta\|y_0\|_\beta^\beta-a_2(T-1),\end{aligned}$$

where the norm $\|\ \|_\beta$ is defined by $\|y\|_\beta=(\sum_{k=1}^{T}|y|^\beta)^{\frac{1}{\beta}}$ for all $y\in H$, $\beta>2$. By Lemma 2.1 in [4], we obtain that the norm $\|\ \|_\beta$ and the norm $\|\ \|$ are equivalent. Hence, we choose the constant S independent of y_0 and w, such that

$$I_w(sy_0)\leqslant 0,\quad\text{for all}\quad s\geqslant S.$$

Combining Claims 1 and 2, we prove that all conditions of Lemma 2.2 are satisfied, and the problem (3.1) has at least one solution y_w which can be obtained as a critical point of I_w at an inf max level. Namely,

$$I'_w(y_w)=0,\quad I_w(y_w)=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_w(\gamma(t)),\tag{3.6}$$

where $\Gamma=\{\gamma\in C([0,1];H)|\gamma(0)=0,\gamma(1)=Sy_0\}$. From now on, we fix such a S and y_0.

Secondly, we prove that the solution $y_w\neq 0$ obtained above satisfies

$$c_1\leqslant\|y_w\|\leqslant c_2,$$

where c_1, $c_2>0$ are independent of w. Indeed, since y_w is a solution of the problem (3.1), we have

$$\sum_{k=1}^{T+1}(\Delta y_w(k-1))^2=\sum_{k=1}^{T}f(k,y_w(k),\Delta w(k))y_w(k).$$

By (2.6), we have

$$\sum_{k=1}^{T+1} (\Delta y_w(k-1))^2 \geqslant \lambda_{\min} \|y_w\|^2 . \tag{3.7}$$

By (H1) and (H4), we can deduce that there exists $c_\varepsilon > 0$ such that

$$\sum_{k=1}^{T} f(k, y_w(k), \Delta w(k)) y_w(k) \leqslant \varepsilon \|y_w\|^2 + c_\varepsilon \|y_w\|_{p+1}^{p+1} . \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\lambda_{\min} \|y_w\|^2 \leqslant \varepsilon \|y_w\|^2 + c_\varepsilon \|y_w\|_{p+1}^{p+1} .$$

Choosing $\varepsilon < \lambda_{\min}$, we have $\|y_w\| \geqslant c_1$ since $p+1 > 2$.

On the other hand, by (3.6), we obtain

$$I_w(y_w) \leqslant \max_{s \geqslant 0} I_w(sy_0).$$

For fixed y_0 with $\|y_0\| = 1$, we use (H3) to get the following estimates:

$$\begin{aligned} I_w(sy_0) &\leq \frac{1}{2}\lambda_{\max} s^2 \|y_0\|^2 - a_1 |s|^\beta \sum_{k=1}^{T} |y_0|^\beta - a_2(T-1) \\ &\leq \frac{1}{2}\lambda_{\max} s^2 - a_1 |s|^\beta \|y_0\|_\beta^\beta - a_2(T-1) =: h(s), \end{aligned}$$

whose maximum is achieved at some $\bar{s}_0 > 0$, and the value $h(\bar{s}_0)$ can be taken as c_2 . So we have $\|y_w\| \leqslant c_2$.

Thirdly, we prove the existence of a positive solution of the problem (3.1) (of course the proof of the existence of a negative solution is analogous). Define the function

$$\hat{f}(k,t,\xi) = \begin{cases} f(k,t,\xi), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Of course $\hat{f}$ satisfies (H3) and (H4) only for $t \geqslant 0$. Applying Lemma 2.2 (Mountain Pass Lemma), we obtain a nontrivial solution $y_w \neq 0$ of the following problem

$$\begin{cases} \Delta^2 y(k-1) + \hat{f}(k, y(k), \Delta w(k)) = 0, \quad k \in [1,T] \\ y(0) = y(T+1) = 0. \end{cases} \tag{3.9}$$

Multiplying the problem (3.9) by y_w^-, where $y_w^- = \min\{0, y_w\}$, we have

$$\begin{aligned} 0 &= \sum_{k=1}^{T+1} (\Delta^2 y_w(k-1) + \sum_{k=1}^{T} \hat{f}(k, y_z(k), \Delta w(k)) y_z^-(k) \\ &= \sum_{k=1}^{T+1} (\Delta y_w(k-1) \Delta y_w^-(k-1) + f(k, 0, \Delta w(k)) y_w^-(k)) \\ &\geq -\sum_{k=1}^{T+1} (\Delta y_w(k-1) \Delta y_w^-(k-1)) \geqslant \lambda_{\min} \left\| y_w^- \right\|^2 . \end{aligned}$$

Therefore, we have $y_w^- = 0$. If $y_w(k) = 0$, we have

$$y_w(k+1) + y_w(k-1) = \Delta^2 y_w(k-1) = -f(k, 0, \Delta w(k)) = 0.$$

So $y_w(k\pm 1)=0$. It follows that y_w vanishes identically. Hence y_w is positive.

Now, we prove our main theorem by using iterative technique.

Proof of Theorem 2.3 Construct a sequence $\{y_n(k)\}\subset H$ as solutions of

$$\begin{cases} \Delta^2 y_n(k-1)+f(k,y_n(k),\Delta y_{n-1}(k))=0, \quad k\in[1,T], \\ y_n(0)=y_n(T+1)=0. \end{cases} \tag{3.10}$$

Using $(3.10)_n$ and $(3.10)_{n+1}$, we have

$$\sum_{k=1}^{T+1}[\Delta y_{n+1}(k-1)(\Delta y_{n+1}(k-1)-\Delta y_n(k-1))] \\ =\sum_{k=1}^{T} f(k,y_{n+1}(k),\Delta y_n(k))(y_{n+1}(k)-y_n(k)), \tag{3.11}$$

$$\sum_{k=1}^{T+1}[\Delta y_{n}(k-1)(\Delta y_{n+1}(k-1)-\Delta y_n(k-1))] \\ =\sum_{k=1}^{T} f(k,y_{n}(k),\Delta y_{n-1}(k))(y_{n+1}(k)-y_n(k)). \tag{3.12}$$

Subtracting (3.12) from (3.11) gives

$$\begin{aligned}&\sum_{k=1}^{T+1}(\Delta y_{n+1}(k-1)-\Delta y_n(k-1))^2 \\ &=\sum_{k=1}^{T}(f(k,y_{n+1}(k),\Delta y_n(k))-f(k,y_n(k),\Delta y_n(k)))(y_{n+1}(k)-y_n(k))+ \\ &\quad\sum_{k=1}^{T}(f(k,y_n(k),\Delta y_n(k))-f(k,y_n(k),\Delta y_{n-1}(k)))(y_{n+1}(k)-y_n(k)).\end{aligned}$$

Thus, by (H5), we have

$$\begin{aligned}&\sum_{k=1}^{T+1}(\Delta y_{n+1}(k-1)-\Delta y_n(k-1))^2 \\ &\leqslant L_1\sum_{k=1}^{T}(y_{n+1}(k)-y_n(k))^2\sum_{k=1}^{T}(\Delta y_n(k)-\Delta y_{n-1}(k))(y_{n+1}(k)-y_n(k)).\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$(\Delta y_n(k)-\Delta y_{n-1}(k))(y_{n+1}(k)-y_n(k))\leqslant\frac{1}{2}(\Delta y_n(k)-\Delta y_{n-1}(k))^2+\frac{1}{2}(y_{n+1}(k)-y_n(k))^2.$$

Hence, we have

$$\begin{aligned}\lambda_{\min}\|y_{n+1}-y_n\|^2 &\le \sum_{k=1}^{T+1}(\Delta y_{n+1}(k-1)-\Delta y_n(k-1))^2 \\ &\le L_1\|y_{n+1}-y_n\|^2+\frac{L_2}{2}\lambda_{\max}\|y_n-y_{n-1}\|^2+\frac{L_2}{2}\|y_{n+1}-y_n\|^2.\end{aligned}$$

From which it follows that

$$\lambda_{\min} \|y_{n+1} - y_n\|^2 \leqslant (L_1 + \frac{L_2}{2}) \|y_{n+1} - y_n\|^2 + \frac{L_2}{2} \lambda_{\max} \|y_n - y_{n-1}\|^2 .$$

Therefore, we have

$$\|y_{n+1} - y_n\| \leqslant \Big(\frac{\frac{L_2}{2}\lambda_{\max}}{\lambda_{\min} - (L_1 + \frac{L_2}{2})} \Big)^{\frac{1}{2}} \|y_n - y_{n-1}\| =: \zeta \|y_n - y_{n-1}\| .$$

Since the coefficient ζ is less than 1 and H is a finite dimensional space, it follows that the sequence $\{y_n\}$ strongly converges to some function $y \in H$.

Since $\|y_n\| \geqslant c_1$ for all n, it follows that $y \neq 0$. In this way, we obtain that Problem (P) has a nontrivial solution. Proceeding as in Lemma 3.1, we conclude that Problem (P) has a positive solution and a negative solution.

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