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Weighted Composition Operators between Bergman-Type Spaces

LIU Jing Cheng, ZHANG Xue Jun, LI Ju Xiang
(College of Mathematics and Computer Science, Hunan Normal University, Hunan 410081, China)
(E-mail: liujingcheng11@126.com)

Abstract Let g_1, g_2 be normal functions. For all $0 < p, q < \infty$, the necessary and sufficient conditions for weighted composition operators $T_{\psi,\varphi} : A^p_{g_1} \to A^q_{g_2}$ to be bounded or compact between Bergman type spaces on the unit ball of C^n are given.

Keywords weighted composition operator; Boundedness; compactness; Bergman type spaces.

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1. Introduction

Let B_n denote the unit ball of C^n, dv be the Lebesgue measure on the unit ball B_n normalized so that $v(B_n) = 1$, and $\mathrm{d}\sigma$ be the normalized rotation invariant measure on the boundary ∂B_n of B_n so that $\sigma(\partial B_n) = 1$. The class of all holomorphic functions on B_n is denoted by $H(B_n)$ and H^∞ denotes the class of all bounded holomorphic function on B_n.

A positive continuous function g on $[0, 1)$ is normal, if there are constants $0 < a < b$ such that

$$\frac{g(r)}{(1-r)^a} \text{ is decreasing for } r \in [0,1) \text{ and } \lim_{r\to 1^-} \frac{g(r)}{(1-r)^a} = 0;$$

$$\frac{g(r)}{(1-r)^b} \text{ is increasing for } r \in [0,1) \text{ and } \lim_{r\to 1^-} \frac{g(r)}{(1-r)^b} = \infty.$$

For $0 < p < \infty$, let g be normal on $[0, 1)$. The Bergman type space A^p_g is the space of functions f that are holomorphic on B_n and satisfy

$$\|f\|^p_{g,p} = \int_{B_n} |f(z)|^p g(|z|)^p (1-|z|)^{-1} \mathrm{d}v(z) < \infty.$$

For $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n)$ in C^n, let $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$. Let $\beta(\cdot,\cdot)$ denote the Bergman metric on B_n. The Bergman ball $E(z, r)$ with center $z \in B_n$ and radius $r > 0$ is defined as $E(z, r) = \{w \in B_n : \beta(z, w) < r\}$.

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Let φ: $B_n \to B_n$ be holomorphic and $\psi \in H(B_n)$. The weighted composition operators $T_{\psi,\varphi}$ on $H(B_n)$ is defined as

$$T_{\psi,\varphi}(f) = \psi(f \circ \varphi) \quad (f \in H(B_n)).$$

Recently several authors have studied the boundedness and compactness of composition operator $T_{1,\varphi}$ or weighted composition operators $T_{\psi,\varphi}$ on Hardy spaces, Beraman spaces and Bloch spaces. For example, in [1], Lou and Shi considered the composition operators from Bergman space A^p_α to A^q_β on the bounded symmetric domains in C^n, and characterized the boundedness and compactness of $T_{1,\varphi}$ in terms of Carleson measures for $0 < p \le q < \infty$. In [2], Xu and Liu characterized the boundedness and compactness of $T_{\psi,\varphi}$ from H^p to H^q $(0 < p \le q < \infty)$. This paper will give the necessary and sufficient conditions for $T_{\psi,\varphi} : A^p_{g_1} \to A^q_{g_2}$ to be bounded or compact between Bergman type spaces on the unit ball of C^n for all $p > 0$, $q > 0$ and normal functions g_1, g_2.

We will use the symbol C or C' to denote a positive constant which does not depend on variables z, w and maybe depend on some parameters, not necessarily the same at each occurrence.

2. Some lemmas

Lemma 2.1[3] *For each $r > 0$, there exists a positive constant C such that*

$$C^{-1} \le \frac{1-|z|^2}{1-|w|^2} \le C \text{ and } C^{-1} \le \frac{1-|z|^2}{|1-\langle z,w\rangle|} \le C$$

for all z and w in B_n with $\beta(z,w) < r$.

Lemma 2.2[3] *Suppose $r_1 > 0$, $r_2 > 0$ and $r_3 > 0$. Then there exists a constant $C > 0$ such that*

$$C^{-1} \le \frac{v(E(z,r_1))}{v(E(w,r_2))} \le C$$

for all $z, w \in B_n$ with $\beta(z,w) < r_3$.

Lemma 2.3[3] *Suppose $r > 0$, $0 < p < \infty$. Then there exists a constant $C > 0$ such that*

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^{n+1}} \int_{E(z,r)} |f(w)|^p \mathrm{d}v(w)$$

for all $f \in H(B_n)$ and all $z \in B_n$.

Lemma 2.4[1] *There exists a postive integer N such that for any $r > 0$, there are sequences $\{w_j\}$ in B_n with the following properties:*

(i) $B_n = \bigcup_{j=1}^\infty E(w_j, r)$;

(ii) Each point $z \in B_n$ belongs to at most N of the sets $E(w_j, 2r)$;

(iii) Specially, some sequence $\{w_j\}$ exists such that $w_j \to \partial B_n$ $(j \to \infty)$ and satisfies (i) and (ii).

Lemma 2.5 *Let $r > 0$, $0 < p < \infty$, and g be normal on $[0,1)$. Then there exists a constant*

$C>0$ such that

$$|f(z)|^p \le \frac{C}{(1-|z|^2)^n g^p(|z|)} \int_{E(z,r)} |f(w)|^p g^p(|w|)(1-|w|)^{-1}\mathrm{d}v(w)$$

for all $f\in H(B_n)$ and all $z\in B_n$.

Proof For $w\in E(z,r)$, by the definition of normal function, we have

$$(\frac{1-|z|}{1-|w|})^a \le \frac{g(|z|)}{g(|w|)} \le (\frac{1-|z|}{1-|w|})^b \quad (|z|\le |w|),$$
$$(\frac{1-|z|}{1-|w|})^b \le \frac{g(|z|)}{g(|w|)} \le (\frac{1-|z|}{1-|w|})^a \quad (|z|> |w|).$$

By Lemma 2.1, there exists a constant $C'>0$ such that $C'^{-1}g(|z|)\le g(|w|)\le C'g(|z|)$ for $w\in E(z,r)$. Then the desired results can be obtained by using Lemma 2.3.

Lemma 2.6[4] *Let $0<q<p<\infty$, $r>0$. If μ is a positive Borel measure on B_n, and g is normal on $[0,1)$, then the following conditions are equivalent:*

(i) $\int_{B_n}\hat{\mu}^s(|z|)(1-|z|)^{-1}g^p(|z|)\mathrm{d}v(z)<\infty$ where $s=\frac{p}{p-q}$, $\hat{\mu}=\frac{\mu(E(z,r))}{(1-|z|^2)^n g^p(z)}$;

(ii) There exists a constant $C>0$ such that

$$\Big\{\int_{B_n}|f(z)|^q\mathrm{d}\mu(z)\Big\}^{\frac{1}{q}} \le C\Big\{\int_{B_n}|f(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\Big\}^{\frac{1}{p}}$$

for all $f\in A^p_g$.

Lemma 2.7 *Let $0<p<\infty$, $\psi\in A^p_g$, φ be a holomorphic self-map of B_n. If f is a nonegative Lebesgue measurable function on B_n, then*

$$\int_{B_n} f(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z)=\int_{B_n} f(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)$$

where $\mu_{\varphi,\psi,g,p}(A)=\int_{\varphi^{-1}(A)}|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)$ for a given Borel set $A\subset B_n$.

Proof First assume that f is a nonegative simple Lebesgue measureable function. Let $f(z)=\sum_{i=1}^m a_i\chi_{A_i}$. Then

$$\begin{aligned}\int_{B_n} f(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z) &= \sum_{i=1}^m a_i\mu_{\varphi,\psi,g,p}(A_i)=\sum_{i=1}^m a_i\int_{A_i}\mathrm{d}\mu_{\varphi,\psi,g,p}(z)\\ &=\sum_{i=1}^m a_i\int_{\varphi^{-1}(A_i)}|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\\ &=\int_{B_n}|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}(\sum_{i=1}^m a_i\chi_{\varphi^{-1}(A_i)\cap B_n})\mathrm{d}v(z)\\ &=\int_{B_n}(f(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z).\end{aligned}$$

If f is a nonegative Lebesgue measurable function, then there exists a monotone increasing simple measurable function sequence $\{f_j\}$, such that $f_j(z)\to f(z)$ $(j\to\infty)$, $z\in B_n$. Therefore,

$$\int_{B_n} f_j(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z)\to\int_{B_n} f(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z)\quad (j\to\infty),$$

and $\{f_j(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\}$ is a monotone increasing simple measurable function sequence. Moreover

$$\{f_j(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\} \to \{f(\varphi(z)|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\} \ \ (j\to\infty),\ z\in B_n.$$

We have

$$\begin{aligned}\int_{B_n} f(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z) &= \lim_{j\to\infty}\int_{B_n} f_j(z)\mathrm{d}\mu_{\varphi,\psi,g,p}(z)\\ &= \lim_{j\to\infty}\int_{B_n} f_j(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\\ &= \int_{B_n} f(\varphi(z))|\psi(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z).\end{aligned}$$

This completes the proof of the lemma. □

3. Boundedness

Theorem 3.1 *Let $0<p<\infty$, $r>0$, $1\le\eta<\infty$. If μ is a positive Borel measure on B_n, and g is normal on $[0,1)$, then the following conditions are equivalent:*

(i) $\sup_{z\in B_n}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^n g^p(|z|)}<\infty$;

(ii) There exists a constant $C>0$ such that

$$\Big\{\int_{B_n}|f(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\le C\int_{B_n}|f(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)$$

for all $f\in A^p_g$.

Proof (i)⟹(ii). Suppose $\sup_{z\in B_n}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^n g^p(|z|)}=M$. Then choosing the sequence $\{w_j\}$ in Lemma 2.4 gives

$$\begin{aligned}\Big\{\int_{B_n}|f(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}} &\le \Big\{\sum_{j=1}^{\infty}\int_{E(w_j,r)}|f(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\ &\le \Big\{\sum_{j=1}^{\infty}\sup\{|f(z)|^{\eta p}: z\in E(w_j,r)\}\mu(E(w_j,r))\Big\}^{\frac{1}{\eta}},\end{aligned}$$

by Lemmas 2.1, 2.5 and the normality of g we get

$$\sup\{|f(z)|^{\eta p}: z\in E(w_j,r)\}\le\Big\{\frac{C}{(1-|w_j|^2)^n g^p(|w_j|)}\int_{E(w_j,2r)}|f(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\Big\}^{\eta}.$$

Thus, by Lemma 2.4 we have

$$\begin{aligned}\Big\{\int_{B_n}|f(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}} &\le \sum_{j=1}^{\infty}\frac{C\mu^{\frac{1}{\eta}}(E(w_j,r))}{(1-|w_j|^2)^n g^p(|w_j|)}\int_{E(w_j,2r)}|f(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\\ &\le MNC\int_{B_n}|f(z)|^p\frac{g^p(|z|)}{1-|z|}\mathrm{d}v(z).\end{aligned}$$

(ii)⟹(i). Assume $f\in A^p_g$ and

$$\Big\{\int_{B_n}|f(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\le C\int_{B_n}|f(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z).$$

Let $f_a(z)=\{\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)(1-<z,a>)^{n+1+\beta}}\}^{\frac{1}{p}}$, where $\beta>pb-1$, $a\in B_n$. By the definition of nomoral function and Lemma 2.1 of [7], we have

$$\begin{aligned}||f_a||_{A_g^p}&=\int_{B_n}|f_a(z)|^p\frac{g^p(|z|)}{1-|z|}\mathrm{d}v(z)=\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)}\int_{B_n}\frac{g^p(|z|)}{|1-\langle z,a\rangle|^{n+1+\beta}(1-|z|)}\mathrm{d}v(z)\\
&\le C\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)}\int_0^1\frac{g^p(r)}{1-r}\int_{\partial B}\frac{\mathrm{d}\sigma(\xi)}{|1-\langle r\xi,a\rangle|^{n+1+\beta}}\mathrm{d}r\\
&\le C'\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)}\int_0^1\frac{g^p(r)}{(1-r)(1-r|a|)^{1+\beta}}\mathrm{d}r\\
&\le C'\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)}\Big\{\int_0^{|a|}+\int_{|a|}^1\Big\}\frac{g^p(r)}{(1-r)(1-r|a|)^{1+\beta}}\mathrm{d}r\\
&\le C'\frac{(1-|a|^2)^{\beta+1}}{g^p(|a|)}\Big\{\frac{g^p(|a|)}{(1-|a|)^{bp}}\int_0^{|a|}\frac{(1-r)^{bp-1}}{(1-r|a|)^{1+\beta}}\mathrm{d}r+\frac{g^p(|a|)}{(1-|a|)^{ap}}\int_{|a|}^1\frac{(1-r)^{ap-1}}{(1-r|a|)^{1+\beta}}\mathrm{d}r\Big\}\\
&\le C''.\end{aligned}$$

Therefore $f_a\in A_g^p$ and $||f_a||_{A_g^p}\le C''$ (C'' is independent of a). Thus

$$\begin{aligned}CC''\ge C||f_a||_{A_g^p}&\ge\Big\{\int_{B_n}\frac{(1-|a|^2)^{(\beta+1)\eta}}{g^{p\eta}(|a|)|1-\langle z,a\rangle|^{(n+1+\beta)\eta}}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\
&\ge\Big\{\int_{E(a,r)}\frac{(1-|a|^2)^{(\beta+1)\eta}}{g^{p\eta}(|a|)|1-\langle z,a\rangle|^{(n+1+\beta)\eta}}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\
&\ge\frac{C'}{(1-|a|^2)^n g^p(|a|)}\Big\{\int_{E(a,r)}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\ge\frac{C'\mu^{\frac{1}{\eta}}(E(a,r))}{(1-|a|^2)^n g^p(|a|)}.\end{aligned}$$

Hence $\sup_{z\in B_n}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^n g^p(|z|)}<\infty$.

This completes the proof of the theorem. □

Theorem 3.2 *Let $0<p,q<\infty$, $\psi\in A_{g_2}^q$, $r>0$. If g_1,g_2 are normal on $[0,1)$, and φ is a holomorphic self-map of B_n, then $T_{\psi,\varphi}$ is a bounded operator from $A_{g_1}^p$ to $A_{g_2}^q$ if and only if:*

(i) When $0<p\le q<\infty$,

$$\sup_{z\in B_n}\frac{\mu_{\psi,\varphi,q,g_2}^{p/q}(E(z,r))}{(1-|z|^2)^n g_1^p(|z|)}<\infty.$$

(ii) When $0<q<p<\infty$,

$$\int_{B_n}\hat{\mu}^s(|z|)(1-|z|)^{-1}g_1^p(|z|)\mathrm{d}v(z)<\infty,$$

where $s=\frac{p}{p-q}$, $\hat{\mu}=\frac{\mu_{\psi,\varphi,q,g_2}(E(z,r))}{(1-|z|^2)^n g_1^p(z)}$, $\mu_{\psi,\varphi,q,g_2}(A)=\int_{\varphi^{-1}(A)}\frac{|\psi(z)|^q g_2^q(|z|)}{1-|z|}\mathrm{d}v(z)$, $A\subset B_n$.

Proof (i) When $0<p\le q<\infty$, if

$$\sup_{z\in B_n}\frac{\mu_{\psi,\varphi,q,g_2}^{p/q}(E(z,r))}{(1-|z|^2)^n g_1^p(|z|)}<\infty$$

From Theorem 3.1, we can find a constant C such that

$$\Big\{\int_{B_n}|f(z)|^q\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z)\Big\}^{\frac{p}{q}}\le C\int_{B_n}|f(z)|^pg_1^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)$$

for all $f\in A^p_{g_1}$.

By the definition of μ_{ψ,φ,q,g_2} and Lemma 2.7, we have

$$\begin{aligned}\|\psi f\circ\varphi\|_{A^q_{g_2}}&=\Big\{\int_{B_n}|\psi(z)|^q|f(\varphi(z))|^qg_2^q(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\Big\}^{\frac{1}{q}}\\&=\Big\{\int_{B_n}|f(z)|^q\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z)\Big\}^{\frac{1}{q}}\le C^{\frac{1}{p}}||f||_{A^p_{g_1}}.\end{aligned}$$

This proves the operator $T_{\psi,\varphi}$ is a bounded operator from $A^p_{g_1}$ to $A^q_{g_2}$.

If $T_{\psi,\varphi}$ is a bounded operator from $A^p_{g_1}$ to $A^q_{g_2}$, then there exists a constant C such that

$$\Big\{\int_{B_n}|\psi(z)|^q|f(\varphi(z))|^qg_2^q(|z|)\mathrm{d}v(z)\Big\}^{\frac{p}{q}}\le C^p\int_{B_n}|f(z)|^pg_1^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)$$

for all $f\in A^p_{g_1}$. By the definition of μ_{ψ,φ,q,g_2} and Lemma 2.7, we have

$$\Big\{\int_{B_n}|f(z)|^q\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z)\Big\}^{\frac{p}{q}}\le C^p\int_{B_n}|f(z)|^pg_1^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z).$$

Using the Theorem 3.1 again, we get

$$\sup_{z\in B_n}\frac{\mu^{p/q}_{\psi,\varphi,q,g_2}(E(z,r))}{(1-|z|^2)^ng_1^p(|z|)}<\infty.$$

(ii) When $0<q<p<\infty$, we can prove the results similar to the proof of case (i) by using Lemmas 2.6 and 2.7.

4. Compactness

Theorem 4.1 *Let $0<p<\infty$, $r>0$, $1\le\eta<\infty$. If μ is a positive Borel measure on B_n, and g is normal on $[0,1)$, then the following conditions are equivalent:*

(i) $\lim_{|z|\to1}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^ng^p(|z|)}=0$;

(ii) $\{\int_{B_n}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\}^{\frac{1}{\eta}}\to0$ $(m\to\infty)$, *for any bounded sequence $\{f_m\}$ of A^p_g converge to 0 uniformly on compact subsets of B_n.*

Proof (i)⇒(ii). Suppose $\lim_{|z|\to1}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^ng^p(|z|)}=0$ for some fixed $r>0$. Choosing the sequence $\{w_j\}$ satisfying the case (iii) of Lemma 2.4, we have

$$\lim_{j\to\infty}\frac{\mu^{\frac{1}{\eta}}(E(w_j,r))}{(1-|w_j|^2)^ng^p(|w_j|)}=0.$$

Then for any $\epsilon>0$ there exists a positive integer number $j_0>0$, when $j>j_0$, we have

$$\frac{\mu^{\frac{1}{\eta}}(E(w_j,r))}{(1-|w_j|^2)^ng^p(|w_j|)}<\epsilon. \tag{4.1}$$

Suppose $\{f_m\}$ is a bounded sequence of A^p_g which converges to 0 uniformly on compact subsets of B_n. Then there exists a constant $K>0$ such that

$$\int_{B_n}|f_m(z)|^p g^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\le K \tag{4.2}$$

for all m.

Using the method similar to that in the proof of Theorem 3.1, we get

$$\begin{aligned}
\Big\{\int_{B_n}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}} &\le \{\sum_{j=1}^{\infty}\int_{E(w_j,r)}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\}^{\frac{1}{\eta}}\\
&\le \sum_{j=1}^{j_0}\Big\{\int_{E(w_j,r)}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}+\sum_{j=j_0+1}^{\infty}\Big\{\int_{E(w_j,r)}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\
&\le \sum_{j=1}^{j_0}\Big\{\int_{E(w_j,r)}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}+\sum_{j=j_0+1}^{\infty}\frac{C_r\mu^{\frac{1}{\eta}}(E(w_j,r))}{(1-|w_j|^2)^n g^p(|w_j|)}\\
&\quad \int_{E(w_j,2r)}|f_m(z)|^p g^p(z)(1-|z|)^{-1}\mathrm{d}v(z).
\end{aligned}$$

By (4.1), (4.2) and Lemma 2.3, we get

$$\Big\{\int_{B_n}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\le\sum_{j=1}^{j_0}\Big\{\int_{E(w_j,r)}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}+\epsilon KNC_r. \tag{4.3}$$

Since f_m converges to 0 uniformly on $\overline{E(w_j,r)}$ for $(m=1,2,\dots,j_0)$, (4.3) means

$$\Big\{\int_{B_n}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\to 0\ \ (m\to\infty).$$

(ii)⇒(i). If $\{\int_{B_n}|f_m(z)|^{\eta p}\mathrm{d}\mu(z)\}^{\frac{1}{\eta}}\to 0$ $(m\to\infty)$, for any bounded sequence $\{f_m\}$ of A_g^p which converges to 0 uniformly on compact subsets of B_n. Let $\{a_m\}$ be a sequence in B_n such that $|a_m|\to 1$ $(m\to\infty)$ and let $f_m(z)=\{\frac{(1-|a_m|^2)^{\beta+1}}{g^p(|a_m|)(1-\langle z,a_m\rangle)^{n+1+\beta}}\}^{\frac{1}{p}}$ $(\beta>pb-1)$. Then $\{f_m\}\in A_g^p$, $||f_m||_{A_g^p}\le C$ (C is independent of m) and $\{f_m\}$ converges to 0 uniformly on compact subsets of B_n. So

$$\begin{aligned}
0\leftarrow&\Big\{\int_{B_n}\frac{(1-|a_m|^2)^{(\beta+1)\eta}}{g^{p\eta}(|a_m|)|1-\langle z,a_m\rangle|^{(n+1+\beta)\eta}}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\
\ge&\Big\{\int_{E(a_m,r)}\frac{(1-|a_m|^2)^{(\beta+1)\eta}}{g^{p\eta}(|a_m|)|1-\langle z,a_m\rangle|^{(n+1+\beta)\eta}}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\\
\ge&\frac{C_r}{(1-|a_m|^2)^n g^p(|a_m|)}\Big\{\int_{E(a_m,r)}\mathrm{d}\mu(z)\Big\}^{\frac{1}{\eta}}\ge\frac{C_r\mu^{\frac{1}{\eta}}(E(a_m,r))}{(1-|a_m|^2)^n g^p(|a_m|)}.
\end{aligned}$$

This means (i) $\lim_{|z|\to 1}\frac{\mu^{\frac{1}{\eta}}(E(z,r))}{(1-|z|^2)^n g^p(|z|)}=0$.

Theorem 4.2 *Let $0<p,q<\infty$, $\psi\in A_{g_2}^q$, $r>0$. If g_1,g_2 are normal on $[0,1)$, and φ is a holomorphic self-map of B_n, then $T_{\psi,\varphi}$ is a compct operator from $A_{g_1}^p$ to $A_{g_2}^q$ if and only if:*

(i) When $0<p\le q<\infty$

$$\lim_{|z|\to 1}\frac{\mu_{\psi,\varphi,q,g_2}^{p/q}(E(z,r))}{(1-|z|^2)^n g_1^p(|z|)}=0.$$

(ii) When $0 < q < p < \infty$

$$\int_{B_n} \hat{\mu}^s(|z|)(1-|z|)^{-1} g_1^p(|z|)\mathrm{d}v(z) < \infty,$$

where $s = \frac{p}{p-q}$, $\hat{\mu} = \frac{\mu_{\psi,\varphi,q,g_2}(E(z,r))}{(1-|z|^2)^n g_1^p(|z|)}$, $\mu_{\psi,\varphi,q,g_2}(A) = \int_{\varphi^{-1}(A)} \frac{|\psi(z)|^q g_2^q(|z|)}{1-|z|}\mathrm{d}v(z)$, $A \in B_n$.

Proof Case (i) can be easily obtained by the definition of μ_{ψ,φ,q,g_2} and Theorem 4.1, and we omite the details here.

(ii) Suppose $\int_{B_n} \hat{\mu}^s(|z|)(1-|z|)^{-1} g_1^p(|z|)\mathrm{d}v(z) < \infty$ as $0 < q < p < \infty$. Let $\{f_k\}$ be a bounded sequence of $A^p_{g_1}$ which converges to 0 uniformly on compact subsets of B_n. By Lemmas 2.5 and 2.7, we have

$$\begin{aligned}
||T_{\psi,\varphi}(f_k)||^q_{A^q_{g_2}} &= \int_{B_n} |\psi(z)|^q |f_k(\varphi(z))|^q g_2^q(|z|)(1-|z|)^{-1}\mathrm{d}v(z) \\
&= \int_{B_n} |f_k(z)|^q \mathrm{d}\mu_{\psi,\varphi,q,g_2}(z) \\
&\le C\int_{B_n} \Big\{\frac{1}{(1-|z|^2)^n g_1^q(|z|)} \int_{E(z,r)} \frac{|f_k(w)|^q g_1^q(|w|)}{(1-|w|)}\mathrm{d}v(w)\Big\}\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z) \\
&= C\int_{B_n} \Big\{\int_{B_n} \frac{\chi_{E(z,r)}(w)|f_k(w)|^q g_1^q(|w|)}{(1-|z|^2)^n g_1^q(|z|)(1-|w|)}\mathrm{d}v(w)\Big\}\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z).
\end{aligned}$$

Applying Fubini's theorem, using that $\chi_{E(z,r)}(w) = \chi_{E(w,r)}(z)$ $(\forall z, w \in B_n)$, Lemma 2.1 and the proof of Lemma 2.5, we get

$$\begin{aligned}
& C\int_{B_n} \Big\{\int_{B_n} \frac{\chi_{E(z,r)}(w)|f_k(w)|^q g_1^q(|w|)}{(1-|z|^2)^n g_1^q(|z|)(1-|w|)}\mathrm{d}v(w)\Big\}\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z) \\
&= C\int_{B_n} \Big\{\int_{B_n} \frac{\chi_{E(z,r)}(w)}{(1-|z|^2)^n g_1^q(|z|)(1-|w|)^{\frac{p-q}{p}}}\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z)\Big\}\frac{|f_k(w)|^q g_1^q(|w|)}{(1-|w|)^{\frac{q}{p}}}\mathrm{d}v(w) \\
&\le C\int_{B_n} \Big\{\int_{E(w,r)} \frac{1}{(1-|w|^2)^n g_1^q(|w|)(1-|w|)^{\frac{p-q}{p}}}\mathrm{d}\mu_{\psi,\varphi,q,g_2}(z)\Big\}\frac{|f_k(w)|^q g_1^q(|w|)}{(1-|w|)^{\frac{q}{p}}}\mathrm{d}v(w) \\
&= C\int_{B_n} \Big(\frac{\mu_{\psi,\varphi,q,g_2}(E(w,r))}{(1-|w|^2)^n g_1^p(|w|)}\frac{g_1^{p-q}(|w|)}{(1-|w|)^{\frac{p-q}{p}}}\Big)\frac{|f_k(w)|^q g_1^q(|w|)}{(1-|w|)^{\frac{q}{p}}}\mathrm{d}v(w).
\end{aligned}$$

Since $\int_{B_n} \hat{\mu}^s(|z|)(1-|z|)^{-1} g_1^p(|z|)\mathrm{d}v(z) < \infty$ and $\{f_k\}$ is a bounded sequnce of $A^p_{g_1}$, there is a constant $M > 0$ such that

$$\Big\{\int_{B_n} |f_k(z)|^p g_1^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\Big\}^{\frac{q}{p}} \le M; \tag{4.4}$$

$$\Big\{\int_{B_n} (\frac{\mu_{\psi,\varphi,q,g_2}(E(w,r))}{(1-|w|^2)^n g_1^p(|w|)})^{\frac{p}{p-q}}(1-|w|)^{-1} g_1^p(|w|)\mathrm{d}v(w)\Big\}^{\frac{p-q}{p}} \le M. \tag{4.5}$$

For any $\epsilon > 0$, there exists a number $\delta \in (0,1)$ such that

$$\Big\{\int_{B_n - \delta B_n} (\frac{\mu_{\psi,\varphi,q,g_2}(E(w,r))}{(1-|w|^2)^n g_1^p(|w|)})^{\frac{p}{p-q}}(1-|w|)^{-1} g_1^p(|w|)\mathrm{d}v(w)\Big\}^{\frac{p-q}{p}} \le \epsilon. \tag{4.6}$$

Because $\{f_k\}$ converges to 0 uniformly on compact subsets of B_n, for above ϵ, there is a constant

$N > 0$ such that

$$\Big\{\int_{\delta B_n} |f_k(z)|^p g_1^p(|z|)(1-|z|)^{-1}\mathrm{d}v(z)\Big\}^{\frac{q}{p}} < \epsilon \tag{4.7}$$

as $k > N$. So for $k > N$, applying Hölder inequality and (4.4)–(4.7) yields

$$\begin{aligned}
&||T_{\psi,\varphi}(f_k)||^q_{A^q_{g_2}}\\
&\leq C\Big\{\int_{\delta B_n}\big(\frac{\mu_{\psi,\varphi,q,g_2}(E(w,r))}{(1-|w|^2)^n g_1^p(|w|)}\big)^{\frac{p}{p-q}}\frac{g_1^p(|w|)}{1-|w|}\mathrm{d}v(w)\Big\}^{\frac{p-q}{p}}\Big\{\int_{\delta B_n}\frac{|f_k(w)|^p g_1^p(|w|)}{1-|w|}\mathrm{d}v(w)\Big\}^{\frac{q}{p}}+\\
&\quad C\Big\{\int_{B_n-\delta B_n}\big(\frac{\mu_{\psi,\varphi,q,g_2}(E(w,r))}{(1-|w|^2)^n g_1^p(|w|)}\big)^{\frac{p}{p-q}}\frac{g_1^p(|w|)}{1-|w|}\mathrm{d}v(w)\Big\}^{\frac{p-q}{p}}\Big\{\int_{B_n-\delta B_n}\frac{|f_k(w)|^p g_1^p(|w|)}{1-|w|}\mathrm{d}v(w)\Big\}^{\frac{q}{p}}\\
&\leq 2CM\epsilon.
\end{aligned}$$

This means $||T_{\psi,\varphi}(f_k)||^q_{A^q_{g_2}} \to 0$ $(k\to\infty)$. So $T_{\psi,\varphi}: A^p_{g_1}\to A^q_{g_2}$ is a compact operator.

If $T_{\psi,\varphi}: A^p_{g_1}\to A^q_{g_2}$ is a compact operator, then $T_{\psi,\varphi}: A^p_{g_1}\to A^q_{g_2}$ must be a bounded operator, and the results can be obtained from Theorem 3.2 clearly.

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