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Iterative Schemes for a Family of Finite Asymptotically Pseudocontractive Mappings in Banach Spaces

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Abstract Let E be a real Banach space and K be a nonempty closed convex and bounded subset of E. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$, be N uniformly L-Lipschitzian, uniformly asymptotically regular with sequences $\{\varepsilon_n^{(i)}\}$ and asymptotically pseudocontractive mappings with sequences $\{k_n^{(i)}\}$, where $\{k_n^{(i)}\}$ and $\{\varepsilon_n^{(i)}\}$, $i = 1, 2, \ldots, N$, satisfy certain mild conditions. Let a sequence $\{x_n\}$ be generated from $x_1 \in K$ by $z_n := (1-\mu_n)x_n + \mu_n T_n^n x_n,\ x_{n+1} := \lambda_n\theta_n x_1 + [1 - \lambda_n(1 + \theta_n)]x_n + \lambda_n T_n^n z_n$ for all integer $n \geqslant 1$, where $T_n = T_{n(\text{mod } N)}$, and $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\mu_n\}$ are three real sequences in $[0, 1]$ satisfying appropriate conditions. Then $||x_n - T_l x_n|| \to 0$ as $n \to \infty$ for each $l \in \{1, 2, \ldots, N\}$. The results presented in this paper generalize and improve the corresponding results of Chidume and Zegeye[1], Reinermann[10], Rhoades[11] and Schu[13].

Keywords approximated fixed point sequence; uniformly asymptotically regular mapping; asymptotically pseudocontractive mapping.

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1. Introduction and preliminaries

Let E be a real normed linear space and E^* its dual space. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by $J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2, ||x|| = ||f||\},\ x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valved. In the sequel, we shall denote the single-valved normalized duality mapping by j.

Let E be a normed linear space, $\emptyset \neq K \subset E$. A mapping $T : K \to K$ is said to be nonexpansive if for all $x, y \in K$ we have $||Tx - Ty|| \leqslant ||x - y||$. It is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that $\|T^n x - T^n y\| \leqslant L\|x - y\|$ for all integers $n \geqslant 1$ and all $x, y \in K$. It is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geqslant 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leqslant k_n\|x - y\|$ for all integers $n \geqslant 1$ and all $x, y \in K$. Clearly, every nonexpansive mapping is asymptotically nonexpansive with sequence $k_n \equiv 1$, $\forall n \geqslant 1$. There are however, asymptotically nonexpansive mappings which are not nonexpansive[4].

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The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk[3] in 1972 and has been studied by severval authors[5, 11−13, 15].

Let K be a subset of real Banach space E and $T : K \to E$ any mapping. T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ such that $\lim_{n\to\infty} k_n = 1$, and there exists $j(x-y) \in J(x-y)$ such that the inequality $\langle T^n x - T^n y, j(x-y)\rangle \leqslant k_n||x-y||^2$ holds for all integers $n \geqslant 1$ and all $x,\ y \in K$. It is easy to know that every asymptotically nonexpansive mapping is asymptotically pseudocontractive mapping.

The class of asymptotically pseudocontractive mappings was introduced by Schu[14] and has been studied by various authors.

The mapping T is called uniformly asymptotically regular if for each $\varepsilon > 0$ there exists integer $n_0 \in \mathbb{N}$, such that $\|T^{n+1}x - T^n x\| \leqslant \varepsilon$ for all $n \geqslant n_0$ and all $x \in K$ and it is called uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ if $\|T^{n+1}x - T^n x\| \leqslant \varepsilon_n$ for all integers $n \geqslant 1$ and all $x \in K$, where $\varepsilon_n \to 0$ as $n \to \infty$.

A family of mappings $\{T_i\}_{i=1}^N$ is called uniformly asymptotically regular if for each $\varepsilon > 0$ there exists integer $n_0 \in \mathbb{N}$, such that $\max_{1\leqslant i,j\leqslant N} \|T_i^{n+1}x - T_j^n x\| \leqslant \varepsilon$ for all $n \geqslant n_0$ and all $x \in K$ and the mapping family $\{T_i\}_{i=1}^N$ is called uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ if $\max_{1\leqslant i,j\leqslant N}\{\|T_i^{n+1}x - T_j^n x\|\} \leqslant \varepsilon_n$ for all integers $n \geqslant 1$ and all $x \in K$, where $\varepsilon_n \to 0$ as $n \to \infty$.

Let K be a nonempty closed convex and bounded subset of a real Banach space E. A mapping $T : K \to K$ is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2, \tag{1.1}$$

for all $x, y \in K$. It follows from a result of Kato[6] that the inequality (1.1) is equivalent to

$$\|x-y\| \leqslant \|x-y+t((I-T)x-(I-T)y)\| \tag{1.2}$$

for all $x, y \in K$ and all $t > 0$, where I denotes the identity mapping.

A mapping T is called strongly pseudocontractive if for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ and $k \in (0,1)$ such that $\langle Tx - Ty, j(x-y)\rangle \leqslant k||x-y||^2$.

Any sequence $\{x_n\}$ satisfying that $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$ for each $l \in \{1, 2, \dots, N\}$, is called an approximate fixed point sequence for a family mappings $\{T_i\}_{i=1}^N$.

The importance of approximate fixed point sequences is that once a sequence has been constructed and proved to be an appropriate fixed point sequence for a continuous mapping T, convergence of that sequence to a fixed point of T is then generally achieved.

For an asymptotically pseudocontractive self-mapping T of a nonempty closed convex and bounded subset of a Hilbert space H, Schu[13] proved the following theorem:

Theorem S[13] *Let H be a Hilbert space, $K \subset E$ be nonempty closed convex and bounded. Let T be a uniformly L-Lipschitzian and asymptotically pseudocontractive self-mapping of K with $\{k_n\} \subset [1, \infty)$; $\sum(q_n^2 - 1) < \infty$, where $q_n = (2k_n - 1)$ for all $n \geqslant 1$, $\alpha_n,\ \beta_n \in [0, 1]$, $\varepsilon \leqslant \alpha_n \leqslant \beta_n \leqslant b$ for all integers $n \geqslant 1$ and some $\varepsilon > 0$; and some $b \in (0, L^{-1}[(1 + L^2)^{1/2} - 1])$; pick $x_0 \in K$; and define $x_{n+1} := \alpha_n T^n z_n + (1-\alpha_n)x_n$; $z_n = \beta_n T^n x_n + (1-\beta_n)x_n$ for all $n \geqslant 0$.*

Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

In 2003, Chidume and Zegeye[1] constructed an approximate fixed point sequence for the class of asymptotically pseudocontractive mappings in Banach spaces and proved the following theorem:

Theorem CZ[1] *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $T : K \to K$ be a uniformly L-Lipschitzian, uniformly asymptotically regular with sequence $\{\varepsilon_n\}$ and asymptotically pseudocontractive with sequence $\{k_n\}$ such that for $\lambda_n, \theta_n \in (0,1)$, $\forall n \geqslant 0$, and satisfying the conditions: (i) $\lambda_n(1+\theta_n) \leqslant 1$, $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$; (ii) $\theta_n \to 0$, $\frac{\lambda_n}{\theta_n} \to 0$, $(\frac{\theta_{n-1}}{\theta_n} - 1)/\lambda_n\theta_n \to 0$, $\frac{\varepsilon_{n-1}}{\lambda_n\theta_n^2} \to 0$; (iii) $k_{n-1} - k_n = o(\lambda_n\theta_n^2)$; (iv) $k_n - 1 = o(\theta_n)$. Let a sequence $\{x_n\}$ be iteratively generated from $x_1 \in K$*

$$x_{n+1} := \lambda_n\theta_n x_1 + [1 - \lambda_n(1+\theta_n)]x_n + \lambda_n T^n x_n, \ \forall n \geqslant 1, \ n \in \mathbb{N}. \tag{1.3}$$

Then $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$.

In this paper, we introduce a new two-step iteration process as follows:

$$\begin{cases} x_{n+1} := \lambda_n\theta_n x_1 + [1 - \lambda_n(1+\theta_n)]x_n + \lambda_n T_n^n z_n, \\ z_n := (1-\mu_n)x_n + \mu_n T_n^n x_n, \ n \geqslant 1, \end{cases} \tag{1.4}$$

where $\{T_i\}_{i=1}^N : K \to K$, are N asymptotically pseudocontractive mappings, $T_n = T_{n(\mathrm{mod}\,N)}$, $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\mu_n\}$ are three real sequences in $[0,1]$ satisfying $\lambda_n(1+\theta_n) \leq 1$ for all $n \geq 1$ and x_0 is a given point in K.

Especially, if $\{\lambda_n\}$, $\{\theta_n\}$ are two sequences in $[0,1]$ satisfying $\lambda_n(1+\theta_n) \leq 1$ for all $n \geq 1$ and x_0 is a given point in K, then the sequence $\{x_n\}$ is defined by

$$x_{n+1} := \lambda_n\theta_n x_1 + [1 - \lambda_n(1+\theta_n)]x_n + \lambda_n T_n^n x_n, \ \ \forall n \geq 1. \tag{1.5}$$

Remark 1.1 If $T_1 = T_2 = \cdots = T_N = T$ or $N = 1$, then (1.5) reduces to (1.3).

The purpose of this paper is to construct an approximate fixed point sequence for a finite family of asymptotically pseudocontractive mappings $\{T_i\}_{i=1}^N$ in Banach spaces. The results presented in this paper generalize and improve the corresponding results of Chidume and Zegeye[1], Reinermann[10], Rhoades[11] and Schu[13].

In order to prove the main result of this paper, we need the following Lemmas:

Lemma 1.1[2,8] *Let E be a real normed linear space. Then for any $x, y \in E$ and $j(x+y) \in J(x+y)$, we have $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle$.*

Lemma 1.2[7] *Let $\{\rho_n\}$, $\{\sigma_n\}$ and $\{\alpha_n\}$ be three sequences of nonnegative numbers satisfying the conditions $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\frac{\sigma_n}{\alpha_n} \to 0$, as $n \to \infty$. Let the recursive inequality $\rho_{n+1}^2 \leqslant \rho_n^2 - \alpha_n\psi(\rho_{n+1}) + \sigma_n$, $n \geqslant 1$ be given, where $\psi : [0,+\infty) \to [0,+\infty)$ is a strictly increasing function such that it is positive on $(0,+\infty)$ and $\psi(0) = 0$. Then $\rho_n \to 0$ as $n \to \infty$.*

2. Main results

Lemma 2.1 *Let E be a real Banach space, and K be a nonempty closed convex and bounded* *subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N uniformly asymptotically regular, uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with sequences $\{k_n^{(i)}\}$, $i = 1, 2, \ldots, N$. Then for $u \in K$ and $t_n \in (0, 1)$ such that $t_n \to 1$ as $n \to \infty$, there exists a sequence $\{y_n\} \subset K$ satisfying the following condition:*

$$y_n = \frac{t_n}{k_n} T_n^n y_n + \Big(1 - \frac{t_n}{k_n}\Big) u, \tag{2.1}$$

where $k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}$, $T_n = T_{n(\mathrm{mod}\, N)}$. Furthermore, we have $\|y_n - T_n y_n\| \to 0$, as $n \to \infty$.

Proof Since $T_i : K \to K$, $i = 1, 2, \ldots, N$, is uniformly L-Lipschitzian, there exists $L_i > 0$, $i = 1, 2, \ldots, N$ such that $\|T_i^n x - T_i^n y\| \leqslant L_i \|x - y\| \leqslant L\|x - y\|$ for all $n \geqslant 1$ and all $x, y \in K$, where $L = \max\{L_1, L_2, \ldots, L_N\}$.

For each $n \geqslant 1$, define the mapping $S_n : K \to K$ by $S_n(y) := \frac{t_n}{k_n} T_n^n y + (1 - \frac{t_n}{k_n})u$. Then $S_n : K \to K$ is continuous and strongly pseudocontractive. Therefore, by Theorem 5 of Reich[9], S_n has a unique fixed point (say) $y_n \in K$. This means that the equation $y_n = \frac{t_n}{k_n} T_n^n y_n + (1 - \frac{t_n}{k_n})u$ has a unique solution for each $t_n \in (0, 1)$. Moreover, since K is bounded, we have that

$$\begin{aligned} \|y_n - T_n^n y_n\| &= \Big\| \Big(1 - \frac{t_n}{k_n}\Big) u + \Big(\frac{t_n}{k_n} - 1\Big) T_n^n y_n \Big\| \\ &= \Big(1 - \frac{t_n}{k_n}\Big) \|u - T_n^n y_n\| \to 0 \ \text{ as } \ n \to \infty. \end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned} \|y_n - T_n y_n\| &= \Big\| \Big(1 - \frac{t_n}{k_n}\Big)(u - T_n y_n) + \frac{t_n}{k_n}(T_n^n y_n - T_n y_n) \Big\| \\ &\leqslant \Big(1 - \frac{t_n}{k_n}\Big) \|u - T_n y_n\| + \frac{t_n}{k_n} \|T_n^n y_n - T_n^{n+1} y_n\| + \frac{t_n}{k_n} L \|T_n^n y_n - y_n\|. \end{aligned} \tag{2.3}$$

In view of the uniformly asymptotic regularity of $\{T_i\}_{i=1}^N$, it follows from (2.2) and (2.3) that $\|y_n - T_n y_n\| \to 0$ as $n \to \infty$. □

Theorem 2.2 *Let K be a nonempty closed convex and bounded subset of a real Banach space E. Let $\{T_i\}_{i=1}^N : K \to K$ be N uniformly L-Lipschitzian, asymptotically pseudocontractive with sequence $\{k_n^{(i)}\}$, $i = 1, 2, \ldots, N$, and uniformly asymptotically regular with sequence $\{\varepsilon_n\}$. Let $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\mu_n\}$ be three real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\lambda_n(1 + \theta_n) \le 1$, $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$;

(ii) $\theta_n \to 0$, $\frac{\lambda_n}{\theta_n} \to 0$, $\frac{\mu_n}{\theta_n} \to 0$, $\frac{|\frac{\theta_{n-1}}{\theta_n} - 1|}{\lambda_n \theta_n} \to 0$, $\frac{\varepsilon_{n-1}}{\lambda_n \theta_n^2} \to 0$;

(iii) $|k_{n-1} - k_n| = o(\lambda_n \theta_n^2)$;

(iv) $k_n - 1 = o(\theta_n)$.

Where $k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}$. Suppose further that $x_1 \in K$ is any given point and $\{x_n\}$ is the iterative sequence defined by (1.4). Then $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$ for each $l \in \{1, 2, \ldots, N\}$.

Proof Let $\{y_n\}$ denote the sequence defined as in (2.1) with $t_n = \frac{1}{1+\theta_n}$ and $u = x_1$. Then from

(1.4) and Lemma 1.1 we get the following estimates:

$$\begin{aligned}\|x_{n+1}-y_n\|^2 &= \|x_n - y_n - \lambda_n((x_n - T_n^n z_n) + \theta_n(x_n - x_1))\|^2\\ &\leqslant \|x_n-y_n\|^2 - 2\lambda_n\langle (x_n - T_n^n z_n) + \theta_n(x_n-x_1), j(x_{n+1}-y_n)\rangle\\ &= \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +\\ &\quad 2\lambda_n\langle \theta_n(x_{n+1}-x_n) - (x_n - T_n^n z_n) + \theta_n(x_1-y_n), j(x_{n+1}-y_n)\rangle\\ &\leqslant \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +\\ &\quad 2\lambda_n\Big\langle \theta_n(x_{n+1}-x_n) + \Big[\theta_n(x_1-y_n) - \Big(y_n - \frac{1}{k_n}T_n^n y_n\Big)\Big] -\\ &\quad \Big[\Big(x_{n+1} - \frac{1}{k_n}T_n^n x_{n+1}\Big) - \Big(y_n - \frac{1}{k_n}T_n^n y_n\Big)\Big] +\\ &\quad \Big[\Big(x_{n+1} - \frac{1}{k_n}T_n^n x_{n+1}\Big) - (x_n - T_n^n z_n)\Big], j(x_{n+1}-y_n)\Big\rangle. \end{aligned} \tag{2.4}$$

Observe that from the properties of y_n and the asymptotical pseudocontractivity of T_n, we get that

$$\theta_n(x_1-y_n) - \Big(y_n - \frac{1}{k_n}T_n^n y_n\Big) + \Big(1 - \frac{1}{k_n}\Big)x_1 = 0 \tag{2.5}$$

and

$$\Big\langle \Big(x_{n+1} - \frac{1}{k_n}T_n^n x_{n+1}\Big) - \Big(y_n - \frac{1}{k_n}T_n^n y_n\Big), j(x_{n+1}-y_n)\Big\rangle \geqslant 0. \tag{2.6}$$

Combining (2.5), (2.6) and (2.4) we have

$$\begin{aligned}\|x_{n+1}-y_n\|^2 &\leqslant \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +\\ &\quad 2\lambda_n\Big\langle (\theta_n+1)(x_{n+1}-x_n) - \frac{1}{k_n}(T_n^n x_{n+1} - T_n^n z_n) + \frac{k_n-1}{k_n}(T_n^n z_n - x_1), j(x_{n+1}-y_n)\Big\rangle -\\ &\quad 2\lambda_n\Big\langle \Big(x_{n+1} - \frac{1}{k_n}T_n^n x_{n+1}\Big) - \Big(y_n - \frac{1}{k_n}T_n^n y_n\Big), j(x_{n+1}-y_n)\Big\rangle\\ &\leqslant \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +\\ &\quad 2\lambda_n\Big[(\theta_n+1)\|x_{n+1}-x_n\| + \frac{1}{k_n}\|T_n^n z_n - T_n^n x_{n+1}\| + \frac{k_n-1}{k_n}\|T_n^n z_n - x_1\|\Big]\cdot\|x_{n+1}-y_n\|\\ &\leqslant \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +\\ &\quad 2\lambda_n\Big[(2+L)\|x_{n+1}-x_n\| + L\|z_n - x_n\| + \frac{k_n-1}{k_n}(\|T_n^n z_n\| + \|x_1\|)\Big]\cdot\|x_{n+1}-y_n\|. \end{aligned} \tag{2.7}$$

Notice the fact that $x_{n+1}-x_n = \lambda_n\theta_n x_1 - \lambda_n(1+\theta_n)x_n + \lambda_n T_n^n z_n = \lambda_n u_n$ and $z_n - x_n = \mu_n(T_n^n x_n - x_n) = \mu_n v_n$, where $u_n = \theta_n x_1 - (1+\theta_n)x_n + T_n^n z_n$, $v_n = T_n^n x_n - x_n$. Since K is bounded, which implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{T_n^n x_n\}$ and $\{T_n^n z_n\}$ are all bounded, there exists $M_1 > 0$ such that

$$\max\{\|x_{n+1}-y_n\|, \|u_n\|, \|v_n\|, \|T_n^n z_n\| + \|x_1\|\} \leq M_1, \tag{2.8}$$

and so

$$\|x_{n+1}-x_n\| = \lambda_n\|u_n\| \leqslant \lambda_n M_1, \quad \|z_n - x_n\| = \mu_n\|v_n\| \leqslant \mu_n M_1. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we have

$$\|x_{n+1}-y_n\|^2 \leqslant \|x_n-y_n\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 +$$

$$2(2+L)\lambda_n^2 M_1^2 + 2\lambda_n L\mu_n M_1^2 + 2\lambda_n \frac{k_n-1}{k_n} M_1^2. \tag{2.10}$$

Moreover, observe that $\overline{T} := \frac{1}{k_n} T_n^n$ is pseudocontractive. Thus it follows from (1.2) that

$$\begin{aligned}
\|y_{n-1}-y_n\| &\leqslant \left\| y_{n-1}-y_n+\frac{1}{\theta_n}[(I-\overline{T})y_{n-1}-(I-\overline{T})y_n]\right\| \\
&= \left|\left(\frac{\theta_{n-1}}{\theta_n}-1\right)(x_1-y_{n-1})+\frac{1}{\theta_n k_{n-1}}\left(T_{n-1}^{n-1}y_{n-1}-T_n^n y_{n-1}\right)+\frac{1}{\theta_n}\left(\frac{1}{k_{n-1}}-\frac{1}{k_n}\right)\left(T_n^n y_{n-1}-x_1\right)\right\| \\
&\leqslant \left|\frac{\theta_{n-1}}{\theta_n}-1\right|(\|x_1\|+\|y_{n-1}\|)+\frac{\varepsilon_{n-1}}{\theta_n k_{n-1}}+\frac{1}{\theta_n}\frac{|k_n-k_{n-1}|}{k_n k_{n-1}}\Big(\|T_n^n y_{n-1}\|+\|x_1\|\Big).
\end{aligned} \tag{2.11}$$

Because $\{x_n\}$, $\{y_n\}$, $\{T_n^n x_n\}$, $\{T_n^n y_n\}$ and $\{T_n^n y_{n-1}\}$ are bounded, there exists $M_2 > 0$ such that $\max\{2(\|x_n - y_{n-1}\| + \|y_{n-1} - y_n\|), \|x_1\| + \|y_{n-1}\|, \|T_n^n y_{n-1}\| + \|x_1\|\} \leqslant M_2$. Notice that

$$\|x_n-y_n\|^2 \leqslant (\|x_n-y_{n-1}\|+\|y_{n-1}-y_n\|)^2 \leqslant \|x_n-y_{n-1}\|^2+\|y_{n-1}-y_n\|\cdot M_2. \tag{2.12}$$

Combining (2.11), (2.12) and (2.10), we get

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 \leqslant& \|x_n-y_{n-1}\|^2 - 2\lambda_n\theta_n\|x_{n+1}-y_n\|^2 + 2\lambda_n L\mu_n M_1^2 + \\
& 2(2+L)\lambda_n^2 M_1^2 + 2\lambda_n(k_n-1)M_1^2 + \left|\frac{\theta_{n-1}}{\theta_n}-1\right| M_2^2 + \\
& \frac{\varepsilon_{n-1}}{\theta_n k_{n-1}} M_2 + \frac{1}{\theta_n}\frac{|k_n-k_{n-1}|}{k_n k_{n-1}} M_2^2.
\end{aligned} \tag{2.13}$$

Thus by Lemma 1.2 and the conditions (i)-(iv) on $\{\lambda_n\}$, $\{\theta_n\}$, $\{\mu_n\}$, $\{k_n\}$ and $\{\varepsilon_n\}$ we get $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Consequently, $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Next we prove that $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$ for each $l \in \{1, 2, \ldots, N\}$. Indeed, by Lemma 2.1 we have that $\|y_n - T_n y_n\| \to 0$ as $n \to \infty$. Thus

$$\begin{aligned}
\|x_n - T_n x_n\| &\leqslant \|x_n - y_n\| + \|y_n - T_n y_n\| + \|T_n y_n - T_n x_n\| \\
&\leqslant L(1+L)\|x_n - y_n\| + \|y_n - T_n y_n\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{2.14}$$

From the condition $\lambda_n \to 0$ as $n \to \infty$ and (2.9) we have $\|x_{n+1} - x_n\| \leqslant \lambda_n M_1 \to 0$ as $n \to \infty$, and so $\|x_n - x_{n+l}\| \to 0$ as $n \to \infty$ for each $l \in \{1, 2, \ldots, N\}$. Thus, for each $l \in \{1, 2, \ldots, N\}$, from (2.14) we have

$$\begin{aligned}
\|x_n - T_{n+l}x_n\| &\leqslant \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\
&\leqslant (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which implies that the sequence $\bigcup_{l=1}^N \{\|x_n - T_{n+l}x_n\|\}_{n=1}^\infty \to 0$ as $n \to \infty$. For each $l \in \{1, 2, \ldots, N\}$, observe that

$$\begin{aligned}
\{\|x_n - T_l x_n\|\}_{n=1}^\infty &= \{\|x_n - T_{n+(l-n)}x_n\|\}_{n=1}^\infty \\
&= \{\|x_n - T_{n+l_n}x_n\|\}_{n=1}^\infty \subset \bigcup_{l=1}^N \{\|x_n - T_{n+l}x_n\|\}_{n=1}^\infty,
\end{aligned}$$

where $l - n = l_n (\operatorname{mod} N)$, $l_n \in \{1, 2, \ldots, N\}$. Therefore, we have $\|x_n - T_l x_n\| \to 0$ as $n \to \infty$. This completes the proof of Theorem 2.2. □

Remark 2.1 If $\mu_n \equiv 0$ in Theorem 2.2, then $z_n = x_n$, hence we can obtain corresponding results of the iterative process (1.5), which is omitted here.

Remark 2.2 If $T_1 = T_2 = \cdots = T_N = T$ or $N = 1$ in Theorem 2.2, then we can obtain corresponding results, which is omitted here.

Remark 2.3 Theorem 2.2 is a generalization of Theorem CZ, that is, if $\mu_n \equiv 0$ and $T_1 = T_2 = \cdots = T_N = T$ or $N = 1$, then Theorem 2.2 will reduce to Theorem CZ.

Remark 2.4 Theorem 2.2 also improves and extends the corresponding results of Reinermann[10], Rhoades[11] and Schu[13].

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