On Uniqueness of Meromorphic Functions and Their Derivatives in One Angular Domain

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Abstract In this paper, applying the Nevanlinna theory of meromorphic function in one angular domain, we deal with a problem of uniqueness for meromorphic functions and their derivatives sharing three finite value ignoring multiplicities in an angular domain instead of the whole complex plane. Obtained results improve a recent result of Lin Weichuan and Seiki Mori.

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1. Introduction and main results

In this paper, meromorphic function always means a function meromorphic in the whole complex plane. If not otherwise stated, functions $f(z)$ and $g(z)$ in this paper are supposed to be nonconstant. We will use standard notations of the Nevanlinna's value distribution theory[1],[2], such as $T(r, f), \sigma(f)$ to denote, respectively the characteristic function and the order of growth of meromorphic function f. Recall the hyper order of f is defined by

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

We denote by $M(\sigma_2)$ the set of transcendental meromorphic functions of finite hyper order.

For the sake of convenience, we give the following notations and definitions[3]. Let X be a nonempty subset of $\mathbb{C}$. An $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is called an IM (ignoring multiplicities) shared value in X of two functions $f(z)$ and $g(z)$ if in X, $f(z) = a$ if and only if $g(z) = a$ while $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ is called a CM (counting multiplicities) shared value in X if $f(z)$ and $g(z)$ assume a at the same points in X with the same multiplicities. When $X = \mathbb{C}$, it is R. Nevanlinna who proved the first uniqueness theorem, known as the five IM theorem, which says that two functions $f(z)$ and $g(z)$ are identical if they have five IM shared values in $X = \mathbb{C}$. When X is a proper subset of $\mathbb{C}_\infty$, Zheng[4] firstly took into account the uniqueness dealing with five shared

values in some angular domains of $\mathbb{C}$. After that, Zheng[3] first established the five IM theorem in one angular domain instead of the whole complex plane and proved the following theorem.

Therorem A *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 \le \alpha < \beta \le 2\pi$ and for some positive number ε and for some $a \in \mathbb{C}_\infty$*

$$\limsup_{r \to \infty} \frac{\log n(r, X_\varepsilon, f = a)}{\log r} > \omega,$$

where $n(r, X_\varepsilon, f = a)$ is the number of the roots of $f(z) - a = 0$ in $\{|z| < r\} \cap X_\varepsilon$, $X_\varepsilon = \{z : \alpha + \varepsilon < \arg z < \beta - \varepsilon\}$ and $\omega = \frac{\pi}{\beta - \alpha}$. We assume that $f(z)$ and $g(z)$ have five distinct IM shared values $a_j, j = 1, 2, \ldots, 5$ in X. Then $f \equiv g$.

Most recently, one of the authors[5] investigated the uniqueness of two functions $f(z)$ and $g(z)$ dealing with the value/set-sharing conditions in one angular domain instead of the plane $\mathbb{C}$ which extended a result by Lin and Mori[6]. In this paper, we shall continue to investigate the uniqueness of two functions in an angular domain. For the uniqueness of meromorphic function in the whole complex plane, Mues, Steinmetz and Gundersen proved the following theorem.

Theorem B[1] *Let $f(z)$ be a meromorphic function, a_1, a_2 and a_3 be distinct finite values. If a_1, a_2 and a_3 are IM shared values of f and f' in $\mathbb{C}$, then $f \equiv f'$.*

Lin and Mori[7] dealt with Theorem B under certain value-sharing condition in a sector instead of the plane $\mathbb{C}$ and proved the following theorem.

Theorem C *Let $f(z)$ be a meromorphic function of infinite order and $\sigma_2(f) < \infty$. Then there exists a direction $\arg z = \theta$ $(0 \le \theta < 2\pi)$ such that for every small positive number ε $(\varepsilon < \frac{\pi}{2})$, $f(z)$ and $f'(z)$ share at most two distinct finite values in the angular domain $\{z : |\arg z - \theta| < \varepsilon\}$.*

The direction $\arg z = \theta$ in Theorem C is called one SV direction by Lin and Mori[7]. Theorem C only discussed the transcendental meromorphic functions of finite hyper order. In this paper, we shall prove that Theorem C is valid for any transcendental meromorphic functions of infinite order. In order to establish our main results, we recall the following definitions and Lemma 1.

Lemma 1[8] *Let $B(r)$ be a positive and continuous function in $[0, +\infty)$ which satisfies*

$$\limsup_{r \to \infty} \frac{\log B(r)}{\log r} = \infty.$$

Then there exists a continuously differentiable function $\rho(r)$, which satisfies the following conditions.

1) $\rho(r)$ is continuous and nondecreasing for $r \ge r_0 (r_0 > 0)$ and tends to $+\infty$ as $r \to +\infty$.

2) The function $U(r) = r^{\rho(r)} (r \ge r_0)$ satisfies the condition

$$\lim_{r \to +\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}.$$

3) $\limsup_{r \to +\infty} \frac{\log B(r)}{\log U(r)} = 1$.

Lemma 1 is due to Xiong[9].

Definition 1 *We define $\rho(r)$ and $U(r)$ in Lemma 1 by the order and type function of $B(r)$ respectively. For a transcendental meromorphic function $f(z)$ of infinite order, we define its order and type function as the order and type function of $T(r, f)$. We denote by $M(\rho(r))$ the set of all meromorphic functions $f(z)$ in $\mathbb{C}$ such that $\limsup_{r \to +\infty} \frac{\log T(r,f)}{\log U(r)} = 1$.*

Definition 2[10] *Let $H(r)$ be a positive and continuous function in $[0, +\infty)$. Let $\rho(r)$ and $U(r)$ be a pair of real functions satisfying Lemma 1. We say that $H(r)$ is of order less than $\rho(r)$ if $\limsup_{r \to \infty} \frac{\log H(r)}{\log U(r)} < 1$. In order that $H(r)$ is of order less than $\rho(r)$, it is necessary and sufficient that we can find a number μ $(0 < \mu < 1)$ such that $H(r) < U^\mu(r)$, when r is sufficiently large.*

The main purpose of this paper is to prove the following theorems.

Theorem 1 *Let $f(z) \in M(\rho(r))$. Given an angular domain $X = \{z : |\arg z - \theta| < \varepsilon\}$, where $0 \le \theta < 2\pi$ and $0 < \varepsilon < \pi$, and for some $a \in \mathbb{C}$*

$$\limsup_{r \to \infty} \frac{\log n(r, \theta, \frac{\varepsilon}{3}, a)}{\log U(r)} = 1, \tag{1}$$

where $n(r, \theta, \frac{\varepsilon}{3}, a)$ is the number of zeros of $f(z) - a$ in $X_{\frac{\varepsilon}{3}}(r) = \{|z| < r\} \cap \{z : |\arg z - \theta| < \frac{\varepsilon}{3}\}$. Then for every small positive number $\varepsilon(\varepsilon < \frac{\pi}{2})$, $f(z)$ and $f'(z)$ share at most two distinct finite values in the angular domain $\{z : |\arg z - \theta| < \varepsilon\}$.

Remark It is well known that a meromorphic function $f(z) \in M(\rho(r))$ has at least one direction $\arg z = \theta$, $0 \le \theta < 2\pi$ from the origin such that for arbitrary small $0 < \varepsilon < \frac{\pi}{2}$, we have

$$\limsup_{r \to \infty} \frac{\log n(r, \theta, \varepsilon, a)}{\log U(r)} = 1,$$

for all but at most two $a \in \mathbb{C}_\infty$[8]. From Theorem 1, any meromorphic function $g(z) \in M(\rho(r))$ has at least one SV direction. On the other hand, from Theorem 1, we can see that every Borel direction of meromorphic function $f(z)(\in M(\rho(r)))$ is a SV direction of $f(z)$.

In 1992, Frank and Schwick generalized Theorem B and proved the following theorem.

Theorem D[1] *Let $f(z)$ be a meromorphic function, and a_1, a_2 and a_3 be distinct finite values. If a_1, a_2 and a_3 are IM shared values of f and $f^{(k)}$ in $\mathbb{C}$, then $f \equiv f^{(k)}$.*

As the end of this section, we pose the following question: Under the conditions of Theorem 1, Do we have: f and $f^{(k)}$ share at most two distinct finite values in the angular domain $\{z : |\arg z - \theta| < \varepsilon\}$?

2. Some lemmas and the proof of Theorem 1

Our proof requires the Nevanlinna theory in an angular domain. For the sake of convenience, we recall Nevanlinna's notations and definitions as follows. Let $f(z)$ be a meromorphic function. Consider an angular domain $\Omega(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Nevanlinna

defined the following notations[8].

$$A_{\alpha\beta}(r, f) = \frac{k}{\pi} \int_1^r (\frac{1}{t^k} - \frac{t^k}{r^{2k}})\{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\}\frac{\mathrm{d}t}{t};$$

$$B_{\alpha\beta}(r, f) = \frac{2k}{\pi r^k} \int_\alpha^\beta \log^+ |f(te^{i\theta})| \sin k(\theta - \alpha)\mathrm{d}\theta;$$

$$C_{\alpha\beta}(r, f) = 2 \sum_{b_v \in \triangle} (\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}}) \sin k(\beta_v - \alpha),$$

where $k = \frac{\pi}{\beta - \alpha}$, $1 \le r < \infty$ and the summation $\sum_{b_v \in \triangle}$ is taken over all poles $b_v = |b_v|e^{i\theta}$ of the function $f(z)$ in the sector $\triangle : 1 < |z| < r$, $\alpha < \arg z < \beta$. Each pole b_v occurs in the sum $\sum_{b_v \in \triangle}$ as many times as its multiplicity. When pole b_v occurs only once in the sum $\sum_{b_v \in \triangle}$, we denote it by $\overline{C}(r, f)$. Furthermore, for $r > 1$, we define

$$D_{\alpha\beta}(r, f) = A_{\alpha\beta}(r, f) + B_{\alpha\beta}(r, f), \quad S_{\alpha\beta}(r, f) = C_{\alpha\beta}(r, f) + D_{\alpha\beta}(r, f).$$

For the sake of simplicity, we omit the subscripts in all notations and use $A(r, f)$, $B(r, f)$, $C(r, f)$, $D(r, f)$ and $S(r, f)$ instead of $A_{\alpha\beta}(r, f)$, $B_{\alpha\beta}(r, f)$, $C_{\alpha\beta}(r, f)$, $D_{\alpha\beta}(r, f)$ and $S_{\alpha\beta}(r, f)$.

Lemma 2[5,7,8] *Let $f(z)$ be a nonconstant meromorphic function in the plane and $\Omega(\alpha, \beta)$ be an angular domain, where $0 < \beta - \alpha \le 2\pi$. Then,*

   (i) For any value $a \in \mathbb{C}$, we have

$$S(r, \frac{1}{f - a}) = S(r, f) + O(1)$$

holds for any $r > 1$.

   (ii) If $f(z)$ is of finite order, then $Q(r, f) = A(r, \frac{f'}{f}) + B(r, \frac{f'}{f}) = O(1)$.
   If $f(z) \in M(\rho(r))$, then $Q(r, f) = A(r, \frac{f'}{f}) + B(r, \frac{f'}{f}) = O(\log U(r))$.

Lemma 3[7] *Let $f(z)$ be a nonconstant meromorphic function in the complex plane, and a_1, a_2, a_3 be three distinct finite complex numbers. Assume that f and f' share the a_i ($i = 1, 2, 3$) IM in X. Then one of the following two cases holds: (i) $f \equiv f'$, or (ii) $S(r, f) = Q(r, f)$, where $Q(r, f)$ is as defined in Lemma 2.*

   We are now in the position to prove Theorem 1.

Proof Suppose that $f(z)$ and $f'(z)$ share three distinct finite values in the angular domain $\{z : |\arg z - \theta| < \varepsilon\}$. Since $f(z) \in M(\rho(r))$ yields $f \not\equiv f'$, it follows from Lemma 3 that $S(r, f) = Q(r, f)$. By Lemma 2 (ii), we have

$$S(r, f) = O(\log U(r)). \tag{2}$$

We deduce from (2) that the order of $S(r, f)$ is less than that of $\rho(r)$. Thus Definition 2 implies that we can find a number μ ($0 < \mu < 1$) such that

$$S(r, f) < (U(r))^\mu, \tag{3}$$

when r is sufficiently large.

For any $a \in \mathbb{C}$, let $b_v = |b_v|e^{i\beta_v}$ $(v = 1, 2, \ldots)$ be the roots of $f = a$ in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, counting multiplicities. We set $n(r) = n(r, \theta, \frac{\varepsilon}{3}, f = a)$. In the angular domain $\Omega(\theta - \frac{\varepsilon}{3}, \theta + \frac{\varepsilon}{3})$, we have $\theta - \frac{\varepsilon}{3} < \beta_v < \theta + \frac{\varepsilon}{3}$, $v = 1, 2, \ldots$. Hence, we deduce that $\frac{\varepsilon}{6} < \beta_v - \theta + \frac{\varepsilon}{2} < \frac{5\varepsilon}{6}$. It follows from the Lemma 2 (i) that

$$S_{\theta-\varepsilon,\theta+\varepsilon}(R, f) \geq C_{\theta-\varepsilon,\theta+\varepsilon}(R, a) + O(1) \geq C_{\theta-\frac{\varepsilon}{2},\theta+\frac{\varepsilon}{2}}(R, a) + O(1)$$

$$\geq 2 \sum_{1<|b_v|<r,\theta-\frac{\varepsilon}{2}<\beta_v<\theta+\frac{\varepsilon}{2}} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) \sin \frac{\pi}{\varepsilon}(\beta_v - \theta + \frac{\varepsilon}{2}) + O(1)$$

$$\geq 2 \sum_{1<|b_v|<r,\theta-\frac{\varepsilon}{3}<\beta_v<\theta+\frac{\varepsilon}{3}} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) \sin \frac{\pi}{\varepsilon}(\beta_v - \theta + \frac{\varepsilon}{2}) + O(1)$$

$$\geq \sum_{1<|b_v|<r,\theta-\frac{\varepsilon}{3}<\beta_v<\theta+\frac{\varepsilon}{3}} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{R^{2k}}\right) + O(1),$$

where $k = \frac{\pi}{\varepsilon}$ and R is defined in Lemma 1. We write the above sum as a Stieltjes-integral and apply the integration by parts of the Stieltjes-integral

$$S_{\theta-\varepsilon,\theta+\varepsilon}(R, f) \geq \int_1^r \frac{1}{t^k}\mathrm{d}n(t) - \frac{1}{R^{2k}} \int_1^r t^k \mathrm{d}n(t) + O(1)$$

$$\geq k \int_1^r \frac{1}{t^{k+1}}n(t)\mathrm{d}t + \frac{n(r)}{r^k} - \frac{r^k n(r)}{R^{2k}} + \frac{k}{R^{2k}} \int_1^r t^{k-1}n(t)\mathrm{d}t + O(1)$$

$$\geq \frac{n(r)}{r^k} - \frac{r^k n(r)}{R^{2k}} + O(1)$$

$$\geq \frac{n(r)}{r^k} - \frac{R^k n(r)}{R^{2k}} + O(1)$$

$$\geq \left(\frac{1}{r^k} - \frac{1}{R^k}\right)n(r) + O(1). \tag{4}$$

Since

$$\log \frac{1}{\frac{1}{r^k} - \frac{1}{R^k}} = \log \frac{r^k R^k}{R^k - r^k} = \log \frac{r^k R^k}{r^k[(1 + \frac{1}{\log U(r)})^k - 1]} = \log \frac{R^k}{(1 + \frac{1}{\log U(r)})^k - 1}$$

$$= \log \frac{(\log U(r))^k R^k}{(1 + \log U(r))^k - (\log U(r))^k} \leq \log(\log U(r))^k R^k,$$

we have

$$\limsup_{r\to\infty} \frac{\log \frac{1}{\frac{1}{r^k} - \frac{1}{R^k}}}{\log U(r)} \leq \limsup_{r\to\infty} \frac{\log(\log U(r))^k R^k}{\log U(r)} = 0.$$

Therefore, for any $\alpha > 0$ and any sufficiently large r, we have

$$\frac{1}{\frac{1}{r^k} - \frac{1}{R^k}} < U^{\frac{\alpha}{2}}(r).$$

So, we can deduce that

$$\limsup_{r\to\infty} \frac{\frac{1}{\frac{1}{r^k} - \frac{1}{R^k}}}{U^\alpha(r)} = 0. \tag{5}$$

From (3)–(5), we deduce that there exists a number μ' $(0 < \mu' < 1)$ such that for any $a \in \mathbb{C}$,

$$n(r, \theta, \frac{\varepsilon}{3}, f = a) < U^{\mu'}(r),$$

if r is sufficiently large. This contradicts hypothesis (1) and the proof of Theorem 1 is completed.

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